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A robust solution of the generalized polynomial Bezout identity

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This paper has been written in honor of Professor Peter Lancaster in recognition of his many important contributions to the theory of polynomial matrices.

Abstract

In this paper, algorithms for the computation of all matrices of the generalized polynomial Bezout identity are proposed. The algorithms are based on the computation of minimal polynomial basis for the right null spaces of certain polynomial matrices. For this reason, an algorithm for the computation of minimal polynomial bases is also proposed. Since this algorithm relies solely on singular value decompositions of certain real matrices, formed with the coefficients of the polynomial matrix whose minimal polynomial bases one is interested in finding, it can be said to be robust.

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1. Introduction

In dealing with polynomial matrices, a particular problem which is important from both, mathematical [1,2] and system theory [3–5] points of view, is the computation of the solution of the generalized polynomial Bezout identity. This problem can be formulated as follows: giving a matrix $G(s)$ belonging to the ring of rational matrices of dimension $p \times q$ (here denoted as $\mathbb{R}^{p\times q}(s)$) find matrices $N(s)$, $M(s)$, $\widetilde{N}(s)$, $\widetilde{M}(s)$, $X(s)$, $Y(s)$, $\widetilde{X}(s)$ and $\widetilde{Y}(s)$ of appropriate dimensions belonging to the ring of polynomial matrices (here denoted as $\mathbb{R}^{m\times n}[s]$, where m and n may assume the values of p or q when appropriate) such that

$$G(s) = N(s)M^{-1}(s) = \widetilde{M}^{-1}(s)\widetilde{N}(s), \tag{1}$$

and satisfy the generalized Bezout identity

$$\begin{bmatrix} \widetilde{X}(s) & -\widetilde{Y}(s) \\ -\widetilde{N}(s) & \widetilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} = \begin{bmatrix} I_q & O \\ O & I_p \end{bmatrix}, \tag{2}$$

where $I_n, n = p, q$ denotes the identity matrix of order n. Notice that Eq. (2) implies that the polynomial matrices $N(s)$ and $M(s)$ ($\widetilde{N}(s)$ and $\widetilde{M}(s)$) are right (left) coprime. For this reason, they are usually referred, in the literature, to as a doubly coprime matrix fraction description (MFD) for $G(s)$.

The usual way to compute the matrices $M(s)$, $N(s)$, $\widetilde{M}(s)$, $\widetilde{N}(s)$, $X(s)$, $Y(s)$, $\widetilde{X}(s)$ and $\widetilde{Y}(s)$ which satisfy Eqs. (1) and (2) is as follows: (*i*) find right and left coprime MFD for $G(s)$; (*ii*) find polynomial matrices $X_1(s)$, $Y_1(s)$, $\widetilde{X}(s)$ and $\widetilde{Y}(s)$ which satisfy, independently, the Bezout equations $\widetilde{X}(s)M(s) - \widetilde{Y}(s)N(s) = I_q$ and $\widetilde{M}(s)X_1(s) - \widetilde{N}(s)Y_1(s) = I_p$; (*iii*) defining $Q(s) = \widetilde{X}(s)Y_1(s) - \widetilde{Y}(s)X_1(s)$ then it is clear that $X(s) = X_1(s) - N(s)Q(s)$, $Y(s) = Y_1(s) - M(s)Q(s)$ satisfy the generalized Bezout equation (2). The computation of right and left coprime MFD of given rational matrix can be carried out in several ways (see [6] and the references therein), while algorithms for solving the Bezout identity are given in [7–11].

To the authors's knowledge, the problem of solving directly the generalized polynomial Bezout identity in a single step has only been addressed by Fang [12], who presented closed forms for the elements of Eq. (2). The deficiency of Fang's algorithm is that it is based on the placement of all eigenvalues of a state matrix, after state feedback, at the origin, and it is well known that this procedure usually leads to numerical difficulties.

In this paper, the problem of solving the generalized polynomial Bezout identity is revisited. Algorithms for the computation of all elements of Eq. (2) are proposed. The algorithms are based on the computation of minimal polynomial basis [13] for the right null spaces of certain polynomial matrices. For this reason, an algorithm for the computation of minimal polynomial bases is also proposed. Since this algorithm relies solely on singular value decompositions (SVD) of certain real matrices (the convolution matrices formed with the coefficients of the polynomial matrix whose minimal polynomial bases one is interested in finding), it can be said to be robust. It is important to remark that SVD algorithms have recently been used for some standard polynomial computation [14], leading to an algorithm for computing greatest common divisors of polynomials where the coefficients are not exactly known. Both algorithms, [14] and the one introduced in this paper, are based, respectively, on the SVD of appropriate Sylvester and convolution matrices; however the former uses SVD to obtain the size of the allowable perturbation on the polynomial coefficients while the latter deploys SVD to find the dimension of the null space of the Sylvester matrices and also a basis for it. Another application of SVD on polynomial matrices has been given in [6].

This paper is organized as follows. Section 2 presents the necessary background on minimal polynomial bases and, in the sequel, a robust algorithm for the computation of minimal polynomial basis for the null space of polynomial matrices will be proposed. In Section 3, the algorithm proposed in Section 2 will be used to compute the matrices $M(s)$, $N(s)$, $\widetilde{M}(s)$, $\widetilde{N}(s)$, and in Section 4, this algorithm will be slightly modified in order to be used for the computation of $X(s)$, $Y(s)$, $\widetilde{X}(s)$ and $\widetilde{Y}(s)$. Two examples to illustrate the algorithms proposed in the paper are given in Section 5. Finally, conclusions are drawn in Section 6.

2. Minimal polynomial bases: background and a robust algorithm

Assume that a matrix $T(s) \in \mathbb{R}^{m\times n}[s]$ ($m < n$, for simplicity) has the following Smith form:

$$\Sigma_T(s) = \begin{bmatrix} \varepsilon_1(s) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon_2(s) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \varepsilon_m(s) & 0 & \cdots & 0 \end{bmatrix},$$

where $\varepsilon_k(s) \equiv 0$ for $k = m - \nu + 1, \ldots, m$. In this case, the matrix $T(s)$ is said to have a right null space of dimension $\bar{\nu} = n - m + \nu$, i.e., it is always possible to find a set of $\bar{\nu}$ linearly independent polynomial vectors $\underline{f}(s)$, over the field of rational functions, such that $T(s)\underline{f}(s) = \underline{0}$. This leads to the following well known result.

Theorem 1. *Let* $F(s) = \begin{bmatrix} \underline{f}_1(s) & \underline{f}_2(s) & \cdots & \underline{f}_{\bar{\nu}}(s) \end{bmatrix}$, *where* $\deg[\underline{f}_i(s)] = \phi_i$, *be a polynomial matrix such that* $T(s)F(s) = O$. *Then, the following statement are equivalent*:

(1) $F(s)$ *is a minimal polynomial bases for the right null space of* $T(s)$.
(2) $F(s)$ *is column-reduced and irreducible*.
(3) $F(s)$ *has minimal order, i.e.* $\sum_{i=1}^{\bar{\nu}} \phi_i$ *is a minimum*.

Proof. See [3], Theorem 6.5-10, p. 458. □

Theorem 1 above suggests the following way to compute a minimum bases for the right null space of $T(s)$: find a non null polynomial vector $\underline{f}_1(s)$, with least possible degree ϕ_1, such that $T(s)\underline{f}_1(s) = \underline{0}$; in the sequel, find a vector $\underline{f}_2(s)$ satisfying $T(s)\underline{f}_2(s) = \underline{0}$, with smallest possible degree $\phi_2 \geqslant \phi_1$, among all vectors that are linearly independent from $\underline{f}_1(s)$; continuing in this way, $\bar{\nu}$ polynomial vectors $\underline{f}_i(s)$ with degrees $\phi_1 \leqslant \phi_2 \leqslant \cdots \leqslant \phi_{\bar{\nu}}$ will be obtained.

When the dimension of the right null space of a polynomial matrix is known, either because of the problem under consideration or because it has been calculated, then the problem of computing the right null space of the polynomial matrix $T(s)$ can be turned to a problem of computing the right null space of a real coefficient matrix. In order to do so, write $T(s) = T_0 s^{\alpha} + T_1 s^{\alpha-1} + \cdots + T_{\alpha}$ and $\underline{f}_i(s) = \underline{f}_{i0} s^{\phi_i} + \underline{f}_{i1} s^{\phi_i - 1} + \cdots + \underline{f}_{i\phi_i}$, $T_i \in \mathbb{R}^{m\times n}$ and $\underline{f}_{ij} \in \mathbb{R}^n$. Defining

$$C_{\phi_i}(T) = \begin{bmatrix} T_0 & 0 & \cdots & 0 \\ T_1 & T_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_{\alpha} & T_{\alpha-1} & \cdots & T_0 \\ 0 & T_{\alpha} & \cdots & T_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{\alpha} \end{bmatrix} \quad \text{and} \quad \underline{f}^{(i)} = \begin{bmatrix} \underline{f}_{i0} \\ \underline{f}_{i1} \\ \vdots \\ \underline{f}_{i\phi_i} \end{bmatrix}, \tag{3}$$

where $C_{\phi_i}(T) \in \mathbb{R}^{m(\alpha+\phi_i+1)\times n(\phi_i+1)}$ and $\underline{f}^{(i)} \in \mathbb{R}^{n(\phi_i+1)}$, then, it is clear that

$$T(s)\underline{f}_i(s) = \underline{0} \Longleftrightarrow C_{\phi_i}(T)\underline{f}^{(i)} = \underline{0}. \tag{4}$$

The matrix $C_{\phi_i}(T)$, above, is usually referred to as the convolution matrix of $T(s)$ of order ϕ_i. The following result may then be stated.

Theorem 2. *A necessary condition for a polynomial vector $\underline{f}_i(s)$ of degree ϕ_i to be an element of the minimal polynomial bases for the right null space of $T(s)$ is that $C_{\phi_i}(T)$ be rank deficient and the vector $\underline{f}^{(i)}$ formed by stacking all the coefficient vectors of $\underline{f}_i(s)$, as in Eq.* (3), *be in the right null space of $C_{\phi_i}(T)$.*

Theorems 1 and 2 together can be used to generate a robust algorithm for the computation of a minimal polynomial bases for the right null space of a polynomial matrix (if it exists), i.e., the search for a polynomial vector of the bases can be carried out by computing the singular value decomposition of $C_{\phi_i}(T)$, namely $C_{\phi_i}(T) = U_{\phi_i}\Sigma_{\phi_i}V_{\phi_i}^t$. Candidate polynomial vectors to a minimal polynomial bases of the right null space of $T(s)$ will be formed either from the columns of V_{ϕ_i} which are associated with zero singular values of $C_{\phi_i}(T)$ or, if $C_{\phi_i}(T)$ has more columns than rows, from those columns corresponding to the excess of columns. Thus, a matrix $F(s) = \begin{bmatrix} \underline{f}_1(s) & \underline{f}_2(s) & \cdots & \underline{f}_{\bar{\nu}}(s) \end{bmatrix}$, whose columns form a minimal polynomial bases for the right null space of $T(s)$, can be computed as follows:

Algorithm 1. Let $\bar{\nu}$ be the dimension of the right null space of $T(s)$.

STEP 1: Make $i = 1$ and set $\deg[\underline{f}_i(s)] = \phi_i = 0$.
STEP 2: Form the matrix $C_{\phi_i}(T)$ according to Eq. (3) and compute its singular value decomposition $C_{\phi_i}(T) = U_{\phi_i}\Sigma_{\phi_i}V^t_{\phi_i}$.
STEP 3: Let n_{ϕ_i} denote the dimension of the null space of $C_{\phi_i}(T)$, which is given by the number of zero singular values of $C_{\phi_i}(T)$ plus the number of columns in excess.
If $n_{\phi_i} = 0$, set $\phi_i = \phi_i + 1$ and go back to step 3.
If $n_{\phi_i} > 0$, there will be up to n_{ϕ_i} polynomial vectors of degree ϕ_i which can be inserted in the bases. These vectors will be formed from the last n_{ϕ_i} columns of V_{ϕ_i}. When $n_{\phi_i} > 0$ and $i = 1$ one vector $\underline{f}^{(1)}$ will be used to form a polynomial vector of the bases providing at least one of its first n elements is non-zero. Furthermore, since $F(s)$ should be column reduced, then, when $n_{\phi_i} > 1$, other polynomial vectors will also be vectors of the minimal polynomial bases providing the leading coefficient matrix of the polynomial matrix formed with the polynomial vectors, which already belong to the bases, and the one formed from the appropriate column of V_{ϕ_i}, be full rank (this can be done in a robust way by computing the singular values of the leading coefficient matrix). Repeat this step until either all possible vectors formed from V_{ϕ_i} have been checked or the bases has been completed. Set $i = i + 1$, each time a new polynomial vector is added to the bases set.
STEP 4: If $i < \bar{\nu}$ then set $i = i + 1$ and $\phi_i = \phi_i + 1$ and go back to step 3.

Remark 1

(1) The assumption that the dimension of the right null space ($\bar{\nu}$) is known, by no means, hampers the general use of Algorithm 1. A robust way to compute the dimension of the right null space of a polynomial matrix is by deploying the algorithm for column reduction proposed in [6,15]. The use of this algorithm leads to a Smith equivalent column reduced matrix whose number of identically zero columns is equal to the dimension of its right null space. In addition, as will become clear in the following sections, the dimensions of the right null spaces of appropriate polynomial matrices, formed in order to solve Eq. (2), will always be known in advance.
(2) Theorem 2 suggests that the search of the polynomial vector of the bases should be done by increasing one-by-one the polynomial vector degrees. When these vectors form a column reduced matrix, it is immediate to see that they are linearly independent, and thus, according to Theorem 1, these vectors will also belong to the minimal polynomial bases. In addition, due to Theorem 2, the

real coefficient vectors formed by stacking the coefficient vectors of these polynomial vectors must also satisfy Eq. (4). This is what is actually being done in step 3.

3. Computation of the matrices $M(s)$, $N(s)$, $\widetilde{M}(s)$ and $\widetilde{N}(s)$

Let $G(s) \in \mathbb{R}^{p\times q}(s)$ and assume that $G(s)$ is expressed as follows:

$$G(s) = \frac{1}{d(s)} N_G(s), \tag{5}$$

where $N(s) \in \mathbb{R}^{p\times q}[s]$ and $d(s)$ is a polynomial (the least common multiple of all the denominator polynomials of the entries of $G(s)$). In addition, assume that $G(s)$ is proper, i.e., $\lim_{s\to\infty} G(s) = G_\infty$ (constant). A non-coprime left MFD $\tilde{A}^{-1}(s)\widetilde{B}(s)$ for $G(s)$ can be defined as $\tilde{A}(s) = d(s)I_p$ and $\widetilde{B}(s) = N_G(s)$, where I_p denotes the identity matrix of order p. Let $G(s) = N(s)M^{-1}(s)$ be a right coprime MFD for $G(s)$. Then the polynomial matrices $N(s)$ and $M(s)$ must satisfy:

$$\begin{bmatrix} \widetilde{B}(s) & -\tilde{A}(s) \end{bmatrix} \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = O, \tag{6}$$

which shows that all right coprime MFD of $G(s)$ must be generated by the minimal polynomial bases of the right null space of

$$T_1(s) = \begin{bmatrix} \widetilde{B}(s) & -\tilde{A}(s) \end{bmatrix}. \tag{7}$$

This leads to the following result.

Theorem 3. *Let $F(s)$ be a minimal polynomial bases for the right null space of $T_1(s)$, i.e., $T_1(s)F(s) = O$ and assume that the columns of $F(s)$ have column degrees arranged in descending order. Writing*

$$F(s) = \begin{bmatrix} M(s) \\ N(s) \end{bmatrix}, \tag{8}$$

then $M(s)$ and $N(s)$ are right coprime. Moreover, $M(s)$ is column reduced.

Proof. The first part of the proof is a consequence of the fact that $F(s)$, being a minimal polynomial bases, is irreducible, and thus, Smith equivalent to $[I_q \quad O]^{\mathrm{T}}$.

To prove that $M(s)$ is column reduced, notice that since the columns of $F(s)$ define a minimal polynomial bases for the right null space of $T_1(s)$ then $F(s)$ is column reduced and thus, the matrix

$$F_{hc} = \begin{bmatrix} M_{hc} \\ N_{hc} \end{bmatrix}$$

formed with the highest coefficient matrix of $F(s)$ is full rank. Suppose now that M_{hc} is singular. Thus, there exists a unimodular matrix $U(s)$ [6,15] such that the degree of the first column of $M_1(s) = M(s)U(s)$ is smaller than that of the corresponding column of $M(s)$. Two possibilities may occur: (*i*) the degree of the first column of $N_1(s) = N(s)U(s)$ decreases, which implies that there exists another polynomial vector belonging to the bases of degree less than that of the bases, thus contradicting the assumption that $F(s)$ is a minimal polynomial bases; (*ii*) the degree of the first column of $N_1(s)$ is larger than or equal to the degree of the corresponding column of $M_1(s)$, which implies that either the assumption that $F(s)$ is a minimal polynomial bases or the assumption that $G(s)$ is proper is contradicted. Thus, M_{hc} must be non-singular, which completes the proof. □

Consider, now, the computation of a left coprime MFD $\widetilde{M}^{-1}(s)\widetilde{N}(s)$ for $G(s)$. In order to do so, notice, from Eq. (2), that

$$\begin{bmatrix}-\widetilde{N}(s) & \widetilde{M}(s)\end{bmatrix}\begin{bmatrix}M(s)\\N(s)\end{bmatrix} = O \Longrightarrow \begin{bmatrix}-N^{\mathrm{T}}(s) & M^{\mathrm{T}}(s)\end{bmatrix}\begin{bmatrix}\widetilde{M}^{\mathrm{T}}(s)\\\widetilde{N}^{\mathrm{T}}(s)\end{bmatrix} = O, \quad (9)$$

which shows that $\widetilde{M}(s)$ and $\widetilde{N}(s)$ can also be computed by direct application of Algorithm 1 to the polynomial matrix

$$T_2(s) = \begin{bmatrix}-N^{\mathrm{T}}(s) & M^{\mathrm{T}}(s)\end{bmatrix}. \quad (10)$$

4. Computation of $X(s)$, $Y(s)$, $\widetilde{X}(s)$ and $\widetilde{Y}(s)$

Let us consider, initially, the computation of $\widetilde{X}(s)$ and $\widetilde{Y}(s)$. From Eq. (2), one can write:

$$M^{\mathrm{T}}(s)\widetilde{X}^{\mathrm{T}}(s) - N^{\mathrm{T}}(s)\widetilde{Y}^{\mathrm{T}}(s) - I_q = O, \quad (11)$$

or equivalently,

$$\begin{bmatrix}M^{\mathrm{T}}(s) & -N^{\mathrm{T}}(s) & -I_q\end{bmatrix}\begin{bmatrix}\widetilde{X}^{\mathrm{T}}(s)\\\widetilde{Y}^{\mathrm{T}}(s)\\I_q\end{bmatrix} = O. \quad (12)$$

It has already been shown [8] that Eq. (11) will have a solution if and only if it is possible to find $\begin{bmatrix}\widetilde{X}(s) & \widetilde{Y}(s) & C\end{bmatrix}^{\mathrm{T}}$, with C non-singular, which solves Eq. (12). Therefore, the problem of finding $\widetilde{X}(s)$ and $\widetilde{Y}(s)$, solution to Eq. (11), is equivalent to the problem of computing a $(p + 2q) \times q$ polynomial matrix $\widehat{F}(s)$ whose column vectors belong to the right null space of

$$T_3(s) = \begin{bmatrix}M^{\mathrm{T}}(s) & -N^{\mathrm{T}}(s) & -I_q\end{bmatrix} \quad (13)$$

with the restriction that all sub-matrices formed with the last q rows of the coefficient matrices of $\widehat{F}(s)$ must be identically zero, except that of the independent power of s, whose q bottom rows must form a non-singular matrix. For example, if $\widehat{F}(s)$ has degree ϕ, then

$$\widehat{F}(s) = \widehat{F}_0 s^{\phi} + \widehat{F}_1 s^{\phi-1} + \cdots + \widehat{F}_{\phi-1} s + \widehat{F}_{\phi}, \tag{14}$$

where

$$\widehat{F}_i = \begin{cases} \begin{bmatrix} \widehat{F}_i^{(top)} \\ O_{q\times q} \end{bmatrix}, & i = 0, \ldots, \phi - 1, \\ \begin{bmatrix} \widehat{F}_i^{(top)} \\ C \end{bmatrix}, & i = \phi, \end{cases} \tag{15}$$

where C must be a $q \times q$ non-singular matrix. It is not hard to see that such a matrix always exists since $M(s)$ and $N(s)$ are, by construction, right coprime. Furthermore, it is important to remark that the computation of $\widetilde{X}(s)$ and $\widetilde{Y}(s)$ does not require that a minimal polynomial bases for the right null space of $T_3(s)$ be found. On the other hand, the convolution matrix formed from $T_3(s)$ will have a special form, since it must guarantee that $\widehat{F}_i$ has the form given by (15). For this reason this convolution matrix will be denoted by $\widehat{C}_{\phi_i}(T_3)$ and will be referred to as a modified convolution matrix of $T_3(s)$. In order to obtain $\widehat{C}_{\phi_i}(T_3)$, let us define

$$\widehat{T}_3(s) = \begin{bmatrix} M^{\mathrm{T}}(s) & -N^{\mathrm{T}}(s) \end{bmatrix}. \tag{16}$$

Thus the modified convolution matrix of $T_3(s)$ will be formed as follows:

$$\widehat{C}_{\phi_i}(T_3) = \begin{bmatrix} C_{\phi_i}(\widehat{T}_3) & \begin{matrix} O_{(\phi_i+\alpha)q\times q} \\ -I_q \end{matrix} \end{bmatrix}, \tag{17}$$

where α is the degree of $\widehat{T}_3(s)$ and $C_{\phi_i}(\widehat{T}_3)$ is obtained according to (3). Therefore, the q polynomial vectors which satisfy Eq. (12) can also be found by searching over the right null space of $\widehat{C}_{\phi_i}(T_3)$, defined in (17), for vectors whose last q components form full column rank matrices. A slight modification of Algorithm 1 leads to a systematic manner to carry out this search, as follows.

Algorithm 2

STEP 1: Make $i = 1$ and set $\deg[\hat{\underline{f}}_i(s)] = \phi_i = 0$.

STEP 2: Form the matrix $\widehat{C}_{\phi_i}(T_3)$ according to Eq. (17) and compute its singular value decomposition, i.e., $\widehat{C}_{\phi_i}(T_3) = U_{\phi_i}\Sigma_{\phi_i}V_{\phi_i}^t$.

STEP 3: Let n_{ϕ_i} denote the dimension of the null space of $\widehat{C}_{\phi_i}(T_3)$, which is given by the number of zero singular values of $\widehat{C}_{\phi_i}(T_3)$ plus the number of columns in excess.
If $n_{\phi_i} = 0$, set $\phi_i = \phi_i + 1$ and go back to step 2.
If $n_{\phi_i} > 0$, there will be up to n_{ϕ_i} polynomial vectors of degree ϕ_i which can satisfy Eq. (12). These vectors will be formed from the last n_{ϕ_i} columns of V_{ϕ_i}. When $i = 1$, $\hat{\underline{f}}^{(1)}$ will be chosen among the last n_{ϕ_i} columns of V_{ϕ_i} for which at least one of the last q elements is different from zero. When $n_{\phi_i} > 1$, other polynomial vectors will also satisfy Eq. (12) providing the matrix formed with the last q elements of the vectors $\hat{\underline{f}}^{(i)}$, which have already been chosen, and the

last q elements of the vector under consideration, be full column rank. Repeat this step until either all possible vectors formed from V_{ϕ_i} have been checked or the bases has been completed. Set $i = i + 1$, each time a new polynomial vector is added to the bases set.

STEP 4: If $i < q$ then set $i = i + 1$ and $\phi_i = \phi_i + 1$ and go back to step 2. Otherwise assume that $\max(\phi_i) = \phi$ and denote $\widehat{F}(s) = \widehat{F}_0 s^{\phi} + \widehat{F}_1 s^{\phi-1} + \cdots + \widehat{F}_{\phi-1} s + \widehat{F}_{\phi}$. Form, according to Eq. (15), the matrix C with the last q rows of $\widehat{F}_{\phi}$ and compute $F(s) = \widehat{F}(s)C^{-1}$.

Let us, now, consider the computation of $X(s)$ and $Y(s)$. Since $\widetilde{M}(s)$, $\widetilde{N}(s)$, $\widetilde{X}(s)$ and $\widetilde{Y}(s)$ have already been computed, then, according to Eq. (2), $X(s)$ and $Y(s)$ must satisfy:

$$\begin{bmatrix} \widetilde{X}(s) & -\widetilde{Y}(s) \\ -\widetilde{N}(s) & \widetilde{M}(s) \end{bmatrix} \begin{bmatrix} Y(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} O \\ I_p \end{bmatrix}, \tag{18}$$

or, equivalently,

$$\begin{bmatrix} \widetilde{X}(s) & -\widetilde{Y}(s) & O \\ -\widetilde{N}(s) & \widetilde{M}(s) & -I_p \end{bmatrix} \begin{bmatrix} Y(s) \\ X(s) \\ I_p \end{bmatrix} = O. \tag{19}$$

Therefore, defining

$$T_4(s) = \begin{bmatrix} \widetilde{X}(s) & -\widetilde{Y}(s) & O \\ -\widetilde{N}(s) & \widetilde{M}(s) & -I_p \end{bmatrix}, \tag{20}$$

then the problem of computing $X(s)$ and $Y(s)$ is equivalent to that of finding p polynomial vectors belonging to the right null space of $T_4(s)$ such that the $p \times p$ bottom matrix be full rank. This problem is similar to that of computing $\widetilde{X}(s)$ and $\widetilde{Y}(s)$ and, therefore, the same procedure to find the polynomial vectors of the right null space of $T_3(s)$, which satisfy Eq. (12), can now be followed to obtain $X(s)$ and $Y(s)$, which solve Eq. (19). Indeed, defining

$$\widehat{T}_4(s) = \begin{bmatrix} \widetilde{X}(s) & -\widetilde{Y}(s) \\ -\widetilde{N}(s) & \widetilde{M}(s) \end{bmatrix}, \tag{21}$$

then the modified convolution matrix of $T_4(s)$ ($\widehat{C}_{\phi_i}(T_4)$) will be given by:

$$\widehat{C}_{\phi_i}(T_4) = \begin{bmatrix} C_{\phi_i}(\widehat{T}_4) & \begin{matrix} O_{(\phi_i+\alpha)(p+q)\times p} \\ O_{q\times p} \\ -I_p \end{matrix} \end{bmatrix}, \tag{22}$$

where α is the degree of $\widehat{T}_4(s)$ and $C_{\phi_i}(\widehat{T}_4)$ is formed according to (3). Therefore, providing it is possible to find polynomial matrices $X(s)$ and $Y(s)$ which satisfy Eq. (18), then Algorithm 2 can also be used to find p polynomial vectors belonging to the right null space of $T_4(s)$ with the restrictions imposed by Eq. (19). It is well known that such matrices do exist (the usual approach to find them has been described in the introduction). Moreover they are unique, as shown in the following theorem.

Theorem 4. *Assume that the polynomial matrices* $N(s)$, $M(s)$, $\widetilde{N}(s)$, $\widetilde{M}(s)$, $\widetilde{X}(s)$ *and* $\widetilde{Y}(s)$ *which satisfy Eqs.* (9) *and* (11) *are given. Then, the polynomial matrices* $X(s)$ *and* $Y(s)$ *which solve Eq.* (18) *are unique. Furthermore, they are also given as*:

$$X(s) = [I + N(s)\widetilde{Y}(s)]\widetilde{M}^{-1}(s) \text{ and } Y(s) = M(s)\widetilde{Y}(s)\widetilde{M}^{-1}(s). \tag{23}$$

Proof. Since $\widetilde{M}(s)$ and $\widetilde{N}(s)$ are left coprime there exist two polynomial matrices $X_1(s)$ and $Y_1(s)$ such that $\widetilde{M}(s)X_1(s) - \widetilde{N}(s)Y_1(s) = I_p$, and using Eqs. (9) and (12), then Eq. (2) must be modified as follows:

$$\begin{bmatrix} \widetilde{X}(s) & -\widetilde{Y}(s) \\ -\widetilde{N}(s) & \widetilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y_1(s) \\ N(s) & X_1(s) \end{bmatrix} = \begin{bmatrix} I_q & \Delta(s) \\ O & I_p \end{bmatrix}. \tag{24}$$

Eq. (24) implies that $\begin{bmatrix} \widetilde{X}(s) & -\widetilde{Y}(s) \\ -\widetilde{N}(s) & \widetilde{M}(s) \end{bmatrix}$ is unimodular. Thus, the polynomial matrices $X(s)$ and $Y(s)$ which solve Eq. (18) are uniquely determined as:

$$\begin{bmatrix} Y(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} \widetilde{X}(s) & -\widetilde{Y}(s) \\ -\widetilde{N}(s) & \widetilde{M}(s) \end{bmatrix}^{-1} \begin{bmatrix} O \\ I_p \end{bmatrix}. \tag{25}$$

Suppose, now, that $X(s)$ and $Y(s)$ has been computed in accordance with Eq. (25). Then, $X(s)$ and $Y(s)$ should also satisfy the generalized Bezout identity (2). Rewriting Eq. (2) in reversed order, i.e.,

$$\begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} \begin{bmatrix} \widetilde{X}(s) & -\widetilde{Y}(s) \\ -\widetilde{N}(s) & \widetilde{M}(s) \end{bmatrix} = \begin{bmatrix} I_q & O \\ O & I_p \end{bmatrix},$$

then the following relationships can be obtained:

$$-M(s)\widetilde{Y}(s) + Y(s)\widetilde{M}(s) = O \text{ and } - N(s)\widetilde{Y}(s) + X(s)\widetilde{M}(s) = I_p,$$

which, after some straightforward manipulation, leads to Eq. (23). □

Remark 2. Theorem 4 has not only proved the existence and uniqueness of the polynomial matrices $X(s)$ and $Y(s)$ which satisfy Eq. (18), but has also provided another way to compute $X(s)$ and $Y(s)$. This is so because, since $\widetilde{M}(s)$ is column reduced, then the algorithm proposed in [15] for the computation of polynomial matrix inverses can be used to easily compute $\widetilde{M}^{-1}(s)$.

5. Examples

In this section, two examples will be presented. In the first example, a 2×2 rational matrix will be used to explain in detail all the steps of the proposed algorithms while in the second one, the algorithm will be applied to 5×3 rational matrix.

5.1. Example 1

With the view to illustrating the results of the paper, consider the following rational 2×2 matrix [16]:

$$G(s) = \frac{1}{d(s)} N_G(s) = \frac{1}{d(s)} \begin{bmatrix} -54.32s & -47s + 2 \\ -48.5s - 1.94 & -42s \end{bmatrix}, \tag{26}$$

where $d(s) = s^2 + s - 2$. Suppose we need to find a doubly coprime matrix fraction description of $G(s)$, i.e. $G(s) = N(s)M^{-1}(s) = \widetilde{M}^{-1}(s)\widetilde{N}(s)$, and the matrices $X(s)$, $Y(s)$, $\widetilde{X}(s)$ and $\widetilde{Y}(s)$ which satisfy the generalized Bezout identity.

Notice that $G(s)$ is already expressed in the form required by Eq. (5). Thus, the matrix $T_1(s)$ defined in Eq. (7) will be given by:

$$T_1(s) = \begin{bmatrix} -54.32s & -47s + 2 & -(s^2 + s - 2) & 0 \\ -48.5s - 1.94 & -42s & 0 & -(s^2 + s - 2) \end{bmatrix}.$$

According to Section 3 and Algorithm 1, the computation of $M(s)$ and $N(s)$ is carried out through the computation of the right null space of $C_{\phi_i}(T_1)$, where ϕ_i varies from 0 up to finding two linearly independent vectors belonging to the right null space of $T_1(s)$. In order to do so, let us set, initially, $\phi_1 = 0$ and form the convolution matrix $C_0(T_1)$. This is a 6×4 matrix having the following singular values: 96.3368, 2.9041, 2.4494, and 0.6809, which are clearly different from zero. Thus its right null space is zero dimensional, which, according to Theorem 2, implies that the minimal polynomial bases for the right null space of $T_1(s)$ has no polynomial vector with zero degree. Following Algorithm 1, the next step is to increase ϕ_i. Accordingly, setting now $\phi_1 = 1$, forming the corresponding convolution matrix $C_1(T_1)$ (an 8×8 dimensional matrix), and computing the singular value decomposition $U_1 \Sigma_1 V_1^{\mathrm{T}}$ of $C_1(T_1)$, then it is possible to see that it has the following singular values: 96.3784, 96.3414, 2.8102, 2.6454, 2.2365, 0.9997, 0.0770 and 0.00000000000000. Since $C_1(T_1)$ has only one zero singular value, then with the column of V_1,

$$\underline{f}^{(1)} = \begin{bmatrix} 0.3840 & -0.4345 & 0 & 0 & -0.3840 & 0.4345 & -0.4345 & -0.3724 \end{bmatrix}^{\mathrm{T}},$$

corresponding to the zero singular value, it is possible to form only one vector belonging to the minimal polynomial bases of $T_1(s)$, being given as:

$$\underline{f}_1(s) = \begin{bmatrix} 0.3840s - 0.3840 \\ -0.4345s + 0.4345 \\ -0.4345 \\ -0.3724 \end{bmatrix}.$$

Thus, according to steps 4 and 5 of Algorithm 1, i and ϕ_i should be increased by one, being now $i = 2$ and $\phi_2 = 2$. It is necessary, then, to form the 10×12 dimensional convolution matrix $C_2(T_1)$ and to compute its singular value decomposition $U_2 \Sigma_2 V_2$. This matrix has the following singular values: 96.3858, 96.3785, 96.3384, 2.8311, 2.8076, 2.5947, 1.8086, 0.9997, 0.0887, 0.00000000000001, which implies that, since $C_2(T_1)$ has two columns more than rows, its right null space is three dimensional. Therefore, there are three candidate vectors to complete the bases, namely the last three columns of V_2. Let us form with the last column vector of V_2,

$$\underline{f}^{(2)} = \begin{bmatrix}-0.1920 & 0.2345 & 0 & 0 & -0.0309 & 0.0523 & -0.5942 \\ & & & -0.5389 & 0.2229 & -0.2868 & 0.2868 & 0.2162\end{bmatrix},$$

the polynomial vector

$$\underline{f}_2(s) = \begin{bmatrix}-0.1920s^2 - 0.0309s + 0.2229 \\ 0.2345s^2 + 0.0523s - 0.2868 \\ -0.5942s + 0.2868 \\ -0.5389s + 0.2162\end{bmatrix}.$$

Therefore, since $\underline{f}^{(2)}$ belongs to the right null space of $C_2(T_1)$, then $\underline{f}_2(s)$ will be an element of the minimal polynomial bases for the right null space of $T_1(s)$ if and only if the matrix

$$F(s) = \begin{bmatrix}\underline{f}_1(s) & \underline{f}_2(s)\end{bmatrix}$$

is column reduced. Indeed, forming the high coefficient matrix of $F(s)$,

$$F_{hc} = \begin{bmatrix}0.3840 & -0.1920 \\ -0.4345 & 0.2345 \\ 0 & 0 \\ 0 & 0\end{bmatrix},$$

it can be checked that it has no zero singular value, being therefore a full column rank matrix. Consequently, the columns of $F(s)$ form a minimal polynomial bases for the right null space of $T_1(s)$. Moreover, by partitioning $F(s)$ appropriately, then a right coprime factorization of $G(s)$ will be given as:

$$M(s) = \begin{bmatrix}0.3840s - 0.3840 & -0.192s^2 - 0.0309s + 0.2229 \\ -0.4345s + 0.4345 & 0.2345s^2 + 0.0523s - 0.2868\end{bmatrix}$$

and

$$N(s) = \begin{bmatrix}-0.4345 & -0.5942s + 0.2868 \\ -0.3724 & -0.5389s + 0.2162\end{bmatrix}.$$

The computation of the left coprime matrices $\widetilde{N}(s)$ and $\widetilde{M}(s)$ is carried out in the same way as for $N(s)$ and $M(s)$; the difference is that it is necessary to find a minimal polynomial bases for the right null space of the matrix $T_2(s)$, defined in Eq. (10). Therefore, with the help of Algorithm 1, we obtain:

$$\widetilde{M}(s) = \begin{bmatrix}-0.3823s + 0.3823 & 0.4369s - 0.4369 \\ 0.1359s - 0.1359 & 0.0147s^2 - 0.1406s + 0.1258\end{bmatrix}$$

and

$$\widetilde{N}(s) = \begin{bmatrix}-0.4238 & -0.3823 \\ -0.7147s + 0.1221 & -0.6189s + 0.1359\end{bmatrix}.$$

Let us, now, consider the computation of $\widetilde{X}(s)$ and $\widetilde{Y}(s)$. This problem is slightly different from the above, i.e., it is not necessary to compute a minimal polynomial bases for the right null space of $T_3(s)$, defined in Eq. (13), being enough to find two polynomial vectors in the right null space of $T_3(s)$ such that: (*i*) $T_3(s)\widehat{F}(s) = O$ and; (*ii*) the matrix formed with the last two rows of $\widehat{F}_\phi$ (defined in Eq. (15)) is non-singular. This search is carried out in accordance with Algorithm 2, as follows. Set $i = 1$ and $\phi_1 = 0$ and form the matrix $\widehat{C}_0(T_3)$ (a 6×6 matrix). Performing the singular value decomposition $U_0 \Sigma_0 V_0^t$ of $\widehat{C}_0(T_3)$, it can be seen that its singular values are 1.5007, 1.0004, 0.7102, 0.4927, 0.0101 and 0.00000000000000, which implies that the last column of V_0 will form the first column of $\widehat{F}(s)$ providing that at least one of its last 2 elements is different from zero. This is actually so, and, thus, the first column of $\widehat{F}(s)$ will have zero degree, being given by:

$$\hat{f}_1(s) = \begin{bmatrix} 0 & 0 & -0.6714 & 0.7402 & 0.0160 & -0.0325 \end{bmatrix}^{\mathrm{T}}.$$

Following Algorithm 2, the next step is to increase i and ϕ_i, i.e. $i = 2$ and $\phi_2 = 1$. The modified convolution matrix $\widehat{C}_1(T_3)$ has the following singular values: 1.5630, 1.1900, 1.0004, 0.7687, 0.6456, 0.3640, 0.0101 and 0. Since $\widehat{C}_1(T_3)$ is an 8×10 matrix, there are three candidate vectors for $\hat{f}_2(s)$. Performing the singular value decomposition $U_1 \Sigma_1 V_1^t$ of $\widehat{C}_1(T_3)$, and taking the last column of V_1, then it is clear that it generates, together with the previously chosen vector, the following 2×2 matrix:

$$C = \begin{bmatrix} 0.0160 & 0.0028 \\ -0.0325 & 0.0302 \end{bmatrix},$$

which is clearly non-singular. Consequently, the second column vector of $\widehat{F}(s)$ will be given by:

$$\hat{f}_2(s) = \begin{bmatrix} -0.5320 \\ -0.4583 \\ -0.0602s + 0.4630 \\ 0.0565s - 0.5338 \\ 0.0028 \\ 0.0302 \end{bmatrix}.$$

The bases is now complete and then, according to step 4 of Algorithm 2, $F(s) = \widehat{F}(s)C^{-1}$. Finally, partitioning $F(s)$ appropriately, we obtain:

$$\widetilde{X}(s) = \begin{bmatrix} -30.1232 & -25.9497 \\ -14.8571 & -12.7987 \end{bmatrix}$$

and

$$\widetilde{Y}(s) = \begin{bmatrix} 3.4090s + 9.1368 & -3.1970s - 8.7549 \\ 1.6814s - 16.1615 & -1.5768s + 18.4703 \end{bmatrix}.$$

For the computation of $X(s)$ and $Y(s)$ the same procedure as above should be followed. The only difference is that now the matrices $\widehat{C}_{\phi_i}(T_4)$, $i = 1, 2$ should be formed according to Eq. (22). Following the steps of Algorithm 2, with T_3 replaced by T_4, leads to the following matrices:

$$X(s) = \begin{bmatrix} 2.6132s - 24.8330 & -13.9018 \\ 2.3700s - 22.5223 & -12.6083 \end{bmatrix}$$

and

$$Y(s) = \begin{bmatrix} 0.8443s^2 - 12.1547s - 10.7858 & -4.4913s + 21.9808 \\ -1.0313s^2 + 14.4766s + 13.6655 & 5.4866s - 24.8750 \end{bmatrix}.$$

5.2. *Example 2*

Consider the following 5×3 rational matrix [17]:

$$G(s) = \frac{1}{d(s)} \left(\begin{bmatrix} 0 & 0 & 0 \\ -0.12 & 1 & 0 \\ 0 & 0 & 0 \\ 4.4190 & 0 & -1.6650 \\ 1.5750 & 0 & -0.0732 \end{bmatrix} s^4 + \begin{bmatrix} -1.5750 & 0 & 0.0732 \\ -0.0739 & 1.5415 & -0.0052 \\ 4.4190 & 0 & -1.6650 \\ 1.6674 & 0.0485 & -1.1574 \\ 6.1213 & -0.2909 & -1.8201 \end{bmatrix} s^3 \right.$$
$$+ \begin{bmatrix} -1.1190 & 0.2909 & -0.0646 \\ -0.5319 & 1.6537 & 0.1570 \\ 1.6674 & 0.0485 & -1.1574 \\ 0.1339 & 0.3279 & -0.0918 \\ 0.3466 & -0.1978 & -0.0977 \end{bmatrix} s^2 + \begin{bmatrix} 1.5409 & 0.2527 & -1.2125 \\ -0.2458 & 0 & 0.1828 \\ 0.1339 & 0.3279 & -0.0918 \\ 0 & 0 & 0 \\ 0.2332 & 0 & -0.0835 \end{bmatrix} s$$
$$\left. + \begin{bmatrix} -0.0816 & 0.3712 & -0.0204 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right),$$

where $d(s) = s^5 + 1.5953s^4 + 1.7572s^3 + 0.1112s^2 + 0.0561s$. The problem considered now is, as in example 1, the computation of a doubly coprime matrix fraction description of $G(s)$, i.e. $G(s) = N(s)M^{-1}(s) = \widetilde{M}^{-1}(s)\widetilde{N}(s)$, and the matrices $X(s)$, $Y(s)$, $\widetilde{X}(s)$ and $\widetilde{Y}(s)$ satisfying the generalized Bezout identity 2.

The algorithms proposed in the paper have been implemented using Matlab, leading to the following matrices:

$$M(s) = \begin{bmatrix} 0 & -0.0140 & 0.1326 \\ 0 & -0.3375 & 0.2029 \\ 0 & -0.0679 & 0.0839 \end{bmatrix} s^2 + \begin{bmatrix} 0 & -0.0776 & 0.1471 \\ 0.6452 & -0.5654 & -0.1503 \\ 0 & -0.2319 & 0.0433 \end{bmatrix} s$$
$$+ \begin{bmatrix} 0.1369 & -0.0875 & 0.1999 \\ 0.0511 & -0.0348 & 0.0560 \\ 0.3822 & -0.2832 & 0.2184 \end{bmatrix},$$

$$N(s)=\begin{bmatrix}0 & 0 & 0\\ 0 & -0.3358 & 0.1870\\ 0 & 0 & 0\\ 0 & 0.0513 & 0.4462\\ 0 & -0.0170 & 0.2027\end{bmatrix}s+\begin{bmatrix}0 & 0.0170 & -0.2027\\ 0.6452 & -0.5392 & -0.1637\\ 0 & 0.0513 & 0.4462\\ 0 & 0 & 0\\ 0 & 0.0581 & 0.5051\end{bmatrix},$$

$$\widetilde{M}(s)=\begin{bmatrix}0 & -0.2447 & -0.5788 & -0.2418 & 0.4617\\ 0 & 0.6490 & -0.2527 & -0.0632 & 0.2323\\ 0 & -0.0994 & -0.2922 & 0.0985 & 0.2463\\ 0 & -0.0161 & 0.4458 & -0.0678 & 0.2895\\ -0.5633 & -0.0644 & -0.0485 & -0.0046 & 0.0883\end{bmatrix}s$$

$$+\begin{bmatrix}0 & 0.1329 & -0.0419 & -0.1143 & 0.0890\\ 0 & 0.1056 & 0.1111 & -0.0460 & 0.0496\\ 0 & 0.0615 & -0.0170 & 0.1172 & 0.2757\\ 0 & 0.0866 & -0.0028 & -0.8087 & 0.1311\\ 0 & 0.0224 & 0.6266 & -0.0485 & -0.5028\end{bmatrix},$$

$$\widetilde{N}(s)=\begin{bmatrix}-0.3119 & -0.2447 & 0.3688\\ 0.0088 & 0.6490 & 0.0882\\ 0.8352 & -0.0994 & -0.1821\\ 0.1585 & -0.0161 & 0.0916\\ 0.1264 & -0.0644 & 0.0012\end{bmatrix},$$

$$X(s)=\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ -0.4658 & -1.4202 & 1.1373 & 5.7122 & -15.7290\\ 0 & 0 & 0 & 0 & 0\\ 0.4644 & 0.2067 & 0.4094 & -1.2546 & 0\\ 0.1364 & -0.0762 & 0.2950 & 0.1342 & -1.7754\end{bmatrix},$$

$$Y(s)=\begin{bmatrix}0.0840 & -0.0618 & 0.2006 & 0.1371 & -1.2858\\ -0.4557 & -1.4276 & 1.1614 & 5.7286 & -15.8833\\ -0.0560 & -0.2881 & 0.2865 & 1.1175 & -3.4127\end{bmatrix}s$$

$$+\begin{bmatrix}-0.5924 & -0.7609 & -1.0040 & 1.1772 & -4.1629\\ 0.3525 & -1.4804 & 0.2379 & 0.4954 & -1.2206\\ -3.2578 & -2.2201 & -0.3374 & 3.8039 & -10.4265\end{bmatrix},$$

$$\widetilde{X}(s)=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\widetilde{Y}(s)=\begin{bmatrix}-19.9990 & -1.5499 & -25.1971 & -5.5880 & 13.7315\\ -24.7947 & 0 & -11.2627 & 0 & 0\\ 2.8522 & 0 & -0.9455 & 0 & 0\end{bmatrix}.$$

For reasons of conciseness, only four decimals have been shown in all polynomial matrices $M(s)$, $N(s)$, $\widetilde{M}(s)$, $\widetilde{N}(s)$, $X(s)$, $Y(s)$, $\widetilde{X}(s)$ and $\widetilde{Y}(s)$ obtained above.

However, if all the decimals are used, then the matrices above satisfy Eq. (2) with an error of 4.3521×10^{-14} in magnitude (i.e. the maximum absolute value of the differences between all the coefficients of the polynomial matrix obtained by performing the multiplication of the left hand side of Eq. (2) and the identity matrix of order eight), showing the accuracy of the algorithm.

Remark 3. For example 2, the matrix $\widetilde{X}(s)$ is identically zero. It is not hard to check that a necessary condition for this to happen is the rational matrix $G(s)$ has no finite zero (i.e. does not loose rank for any finite value of $s \in \mathbb{C}$), since in this case $N(s)$ will be full column rank for all $s \in \mathbb{C}$. Since $N(s)M^{-1}(s)$ and $\widetilde{M}^{-1}(s)\widetilde{N}(s)$ are, respectively, right and left coprime matrix fraction descriptions of $G(s)$, then the zeros of $G(s)$ will be also the values of $s \in \mathbb{C}$ for which $\widetilde{N}(s)$ looses rank. However, $\widetilde{N}(s)$ has only real coefficients and has full column rank, showing the consistency of the result.

6. Concluding remarks

In this paper, robust algorithms for the computation of all matrices of the generalized polynomial Bezout identity have been proposed. The robustness of the algorithms comes from the fact that they rely solely on singular value decompositions of real coefficient matrices. A numerical example illustrates the efficiency of the proposed algorithm. It has been suggested by an anonymous reviewer that the computation of singular value decomposition at each step could be replaced by some sort of recursive algorithms (such as QR-factorization) without sacrificing significantly the accuracy. This will be the subject of future research.

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