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Characteristic locus method robustness improvement through optimal static normalizing pre-compensation

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SUMMARY

Controlled systems designed by using the characteristic locus method can be very sensitive to small perturbations in the plant input and output at the frequencies where the plant transfer matrix is far from normal. In order to improve the closed-loop system robustness, previous work proposes the design of a dynamic normalizing pre-compensator followed by the design of a commutative controller for the pre-compensated plant. The restriction on its structure and the need for approximation by a stable rational transfer matrix were among the limitations of the dynamic pre-compensator. This paper shows that it suffices to design a static pre-compensator that makes the pre-compensated plant as closely as possible to a normal matrix in a frequency band containing the crossover frequency. The pre-compensator is found by solving an optimization problem, whose solution is obtained directly by computing either a singular value decomposition of a real matrix or the spectral decomposition of a symmetric matrix, depending on whether normalization is to be achieved in one or more frequencies. Numerical examples illustrate the theoretical results of the paper. Copyright © 2009 John Wiley & Sons, Ltd.

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KEY WORDS: robustness; multivariable systems; frequency domain; characteristic locus method; pre-compensator design

1. INTRODUCTION

The generalized Nyquist stability criterion [1] is an important tool for stability analysis and design of closed-loop multivariable linear systems since it provides an integrated assessment of closed-loop stability and dynamic performance. The design technique that emerged naturally from the generalized Nyquist stability criterion is the well-known characteristic locus method (CLM) [2–4], and the controllers designed according to the CLM are known as commutative controllers, since they commute with the plant transfer function with respect to multiplication. The commutativity property has the effect of changing a multivariable design in m single-input–single-output (SISO) designs, where m is the number of inputs/outputs of the plant to be controlled. This is carried out through the choice of the controller eigenfunctions with the view to modifying the characteristic loci of the plant in the same manner as SISO controllers modify the Nyquist diagram of the plant in scalar systems. In spite of this simplification, the resulting controller is a full multivariable controller, as opposed to those obtained by sequential loop closing or independent design of each loop using

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SISO controllers as is the case of decentralized control systems [5–12].

Despite being acknowledged as an efficient design technique, the CLM was under serious criticism on account of eigenvalue sensitivity [13], i.e. for plants with frequency responses far from normal at a certain frequency band, the characteristic loci can be very sensitive to perturbations in the plant input and output at these frequencies [14, 15]. In order to improve the robustness of closed-loop systems compensated with commutative controllers, a two-stage-design approach has been proposed in [16], as follows: for an $m\times m$ plant with transfer function $G(s)$, a dynamic normalizing pre-compensator, i.e. a pre-compensator $K_{\mathrm{p}}(s)$ that approximately normalizes the pre-compensated plant $G(\mathrm{j}w)K_{\mathrm{p}}(\mathrm{j}w)$ at the whole frequency range is obtained initially and, in the sequel, a controller, $K_{\mathrm{c}}(s)$, that commutes with $G(s)K_{\mathrm{p}}(s)$ and also internally stabilizes the feedback system [17, 18], is designed. The overall controller transfer function is therefore $K(s)=K_{\mathrm{p}}(s)K_{\mathrm{c}}(s)$. More recently, with the view to considering the sensitivity of the characteristic loci with respect to perturbations in both the plant input and output, the design of a normalizing pre-compensator for $G(s)$ has been proposed through the solution of an optimization problem based on a measure of the misalignment between the input and output principal directions of the plant in each frequency [19]. Although this pre-compensation scheme has led to insensitive open-loop characteristic loci, several constraints have been imposed in the pre-compensator structure.

This paper deals with the design of a pre-compensator with the view to normalizing the plant; therefore, improving the robustness of the closed-loop system compensated in accordance with the CLM. It is shown that it suffices to design a static pre-compensator that makes the pre-compensated plant as closely as possible to a normal matrix at a frequency band containing the crossover frequency; therefore, the need for normalization in the whole frequency range, as done in [16, 19]. The pre-compensator is found by solving an optimization problem, whose solution is obtained directly by computing either a singular value decomposition of a real matrix or the spectral decomposition of a symmetric matrix, depending on whether normalization is to be achieved in one or more frequencies. In addition, different from previous works, the proposed pre-compensator has no constraints in its structure.

This paper is structured as follows: in Section 2 it is shown that it suffices to use a static pre-compensator that normalizes the plant at the vicinity of the crossover frequency; in Section 3 an optimization problem for the design of a static pre-compensator with the view to normalizing the pre-compensated plant at a desired frequency ω_0 is formulated and solved; this optimization problem is also extended to the multi-frequency case in this section; finally, two numerical examples are presented in Section 4 to illustrate the theoretical results presented in the paper.

2. PRELIMINARY RESULTS

Let $G(s)$ and $K(s)$ be the $m\times m$ transfer matrices of the plant and controller, respectively. According to the generalized Nyquist stability criterion [1], the closed-loop system of Figure 1 is stable if and only if the net sum of anticlockwise encirclements of the critical point $-1+\mathrm{j}0$, by the characteristic loci of $G(s)K(s)$, is equal to the number of unstable poles of $G(s)$ and $K(s)$. The use of the generalized Nyquist stability criterion as a design tool is by means of the so-called commutative controllers, that is, a controller $K(s)$ such that $G(s)K(s)=K(s)G(s)$. Commutativity is achieved whenever $G(s)$ and $K(s)$ share the same eigenvector and dual eigenvector frames, in which case, the eigenvalues of the product $G(s)K(s)$ are equal to the product of the eigenvalues of $G(s)$ and $K(s)$. In this regard, closed-loop stability and performance requirements such as tracking, disturbance rejection and good transient response are achieved through the manipulation of the eigenfunctions of the open-loop transfer matrix, i.e. by choosing, in an adequate

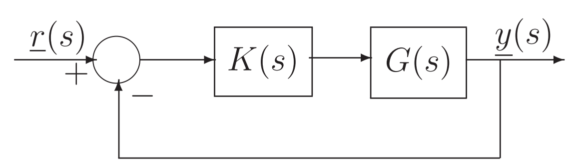

Figure 1. Closed-loop feedback control system.

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manner, the controller eigenfunctions. This controller design strategy is known as the CLM. However, as shown in [13], the CLM can only be effective from the robustness point of view if the plant transfer matrix is normal. The definition of normal matrix is as follows.

Definition 1
A matrix $G \in \mathbb{C}^{m\times m}$ is normal if and only if it commutes with its conjugate transpose, $G^\star$, i.e. $GG^\star = G^\star G$.

Some measures of approximate normality of a complex matrix are presented in [20]. One of these measures is based on the following theorem.

Theorem 1
A matrix $G \in \mathbb{C}^{m\times m}$ is normal if and only if it has an orthonormal set of m eigenvectors.

Proof
See [21]. □

An immediate consequence of Theorem 1 is the definition of the degree of normality of a matrix and of approximately normal matrices. This is done with the help of the following definitions [22].

Definition 2
Let $A \in \mathbb{C}^{m\times m}$. If A^{-1} exists, then the Euclidean condition number of A, denoted as $\mathscr{C}(A)$, is given by

$$\mathscr{C}(A) = \frac{\bar{\sigma}(A)}{\underline{\sigma}(A)} \tag{1}$$

where $\bar{\sigma}(\cdot)$ and $\underline{\sigma}(\cdot)$ denote, respectively, the maximum and minimum singular values of a matrix.

Definition 3
Let $\mathbf{D}^{m\times m}$ be the set of all invertible diagonal matrices in $\mathbb{C}^{m\times m}$. Under the same assumptions as in Definition 2, then the optimal condition number of a complex matrix A, denoted as $\mathscr{C}^{\text{opt}}(A)$, is defined as:

$$\mathscr{C}^{\text{opt}}(A) = \inf_{D \in \mathbf{D}^{m\times m}} \mathscr{C}(AD) \tag{2}$$

It is shown in [22] that optimization problem (2) is convex and, therefore, it has only one minimum that can be achieved through convex programming.

Measures of normality and approximately normality can now be introduced, as follows.

Definition 4
Let $G \in \mathbb{C}^{m\times m}$ and suppose that $G = W\Lambda V$, $V = W^{-1}$, is a spectral decomposition of G. The degree of normality of G is defined as

$$v = \mathscr{C}^{\text{opt}}(W) - 1 \tag{3}$$

Moreover, when $v \to 0$, then the matrix G is said to be approximately normal.

The optimal condition numbers of the eigenvector matrices of $G(\mathrm{j}\omega)K(\mathrm{j}\omega)$ and $K(\mathrm{j}\omega)G(\mathrm{j}\omega)$ can be used to give an upper bound for a measure of robustness of the closed-loop system with respect to perturbations in the plant output and input, respectively. This will be done with the help of the following result.

Lemma 1 (*Small gain theorem*)
Let $M_G(s)$ be a stable transfer matrix and define

$$G_{\mathrm{p}}(s) = [I + M_G(s)]G(s) \tag{4}$$

as the plant transfer matrix with multiplicative perturbation in the plant output, where $\bar{\sigma}[M_G(\mathrm{j}\omega)] \leqslant \mu_G(\omega)$ and $\mu_G(\omega)$ is a nonnegative real function. Then, $K(s)$ stabilizes $G_{\mathrm{p}}(s)$ if and only if $K(s)$ stabilizes $G(s)$ and

$$\bar{\sigma}\{G(\mathrm{j}\omega)K(\mathrm{j}\omega)[I + G(\mathrm{j}\omega)K(\mathrm{j}\omega)]^{-1}\} < \frac{1}{\mu_G(\omega)} \tag{5}$$

Proof
See [23]. □

A necessary condition for robust stability of the closed-loop system, given in terms of the eigenvalues of the open-loop transfer matrix, can be obtained from Lemma 1 and using the fact that the maximum singular value is an upper bound for the moduli of the eigenvalues of a matrix, as follows:

$$\max_{i \in \mathscr{I}_m} \left| \frac{\lambda_i\{G(\mathrm{j}\omega)K(\mathrm{j}\omega)\}}{1 + \lambda_i\{G(\mathrm{j}\omega)K(\mathrm{j}\omega)\}} \right| < \frac{1}{\mu_G(\omega)} \tag{6}$$

where $\mathscr{I}_m = \{1, 2, \ldots, m\}$. Consider now the following result.

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Lemma 2
Let $A \in \mathbb{C}^{m\times m}$ and let $A = W_A \Lambda_A V_A$, $V_A = W_A^{-1}$, be a spectral decomposition of A. Then,

$$\frac{\bar{\sigma}\{A\}}{\mathscr{C}^{\text{opt}}(W_A)} \leqslant \max_{i\in\mathscr{I}_m} |\lambda_i\{A\}| \tag{7}$$

Proof
Note that $A = W_A \Lambda_A V_A = (W_A D)\Lambda_A (D^{-1} V_A)$, for all $D \in \mathbf{D}^{m\times m}$. Thus

$$\begin{aligned}\bar{\sigma}\{A\} = \bar{\sigma}\{W_A D \Lambda_A D^{-1} V_A\} &\leqslant \frac{\bar{\sigma}\{W_A D\}}{\underline{\sigma}\{W_A D\}}\bar{\sigma}\{\Lambda_A\} \\ &= \mathscr{C}(W_A D) \max_{i\in\mathscr{I}_m} |\lambda_i\{A\}| \end{aligned}\tag{8}$$

Replacing D, in inequality (8), with the optimal solution of problem (2) gives the desired result. □

A sufficient condition for robust stability of the closed-loop system, expressed in terms of the eigenvalues of the open-loop transfer matrix, can be obtained with the help of Lemmas 1 and 2, as follows.

Theorem 2
If $K(s)$ stabilizes $G(s)$ and for each $\omega \in \mathbb{R}$

$$\max_{i\in\{1,2,\ldots,m\}} \left| \frac{\lambda_i\{G(\mathrm{j}w)K(\mathrm{j}w)\}}{1+\lambda_i\{G(\mathrm{j}w)K(\mathrm{j}w)\}} \right| < \frac{1}{\mu_G(\omega)[1+v(\mathrm{j}w)]} \tag{9}$$

where $v(\mathrm{j}w) = \mathscr{C}^{\text{opt}}[W(\mathrm{j}w)] - 1$ with $W(\mathrm{j}w)$ denoting the eigenvector matrix of $G(\mathrm{j}w)K(\mathrm{j}w)$, then $K(s)$ stabilizes $G_{\mathrm{p}}(s)$.

Proof
The proof is straightforward and will be omitted. □

Remark 1

(i) Similar results to those given in Theorem 2 and inequality (6) can be obtained assuming multiplicative perturbation in the plant input, i.e. $G_{\mathrm{p}}(s) = G(s)[I + M_G(s)]$. In this case the upper bounds are given in terms of the optimal condition number of the eigenvector matrix of $K(\mathrm{j}w)G(\mathrm{j}w)$.

(ii) When the matrix $G(s)$ is normal at a certain frequency ω, then $v(\mathrm{j}w)$ becomes zero and thus inequalities (9) and (6) become identical. In this case, inequality (6) provides a necessary and sufficient condition, as claimed in [13].

Based on Lemma 1, Theorem 2 and inequality (6), the following conclusions regarding the normality of the open-loop transfer matrix can be drawn

(1) At high frequencies, since $G(s)$ is usually strictly proper, $G(\mathrm{j}w)K(\mathrm{j}w) \to O$. As a consequence, $\lambda_i[G(\mathrm{j}w)K(\mathrm{j}w)] \to 0$, $i = 1, \ldots, m$. Therefore, conditions (6) and (9), become, respectively,

$$\varepsilon < \frac{1}{\mu_G} \quad \text{and} \quad \varepsilon < \frac{1}{\mu_G(1+v)}, \ \varepsilon \to 0 \tag{10}$$

which are satisfied for arbitrarily large values of v. This implies that there is no need for normalization at high frequencies.

(2) At low frequencies, due to integral action required to track step reference signals, $G(\mathrm{j}w)K(\mathrm{j}w)$ becomes infinite and thus, according to inequality (5), $\mu_G < 1$ becomes a necessary and sufficient condition for robust stability of the closed-loop system. Therefore, normalization is not a crucial requirement for low sensitivity of the characteristic loci at low frequencies when integral action is deployed; see also [3].

(3) At the frequencies near the crossover frequency ω_{c}, the sufficient condition (9) approaches the necessary condition (6) when $v \to 0$, namely that when $G(\mathrm{j}w_{\mathrm{c}})K(\mathrm{j}w_{\mathrm{c}})$ becomes approximately normal.

(4) Same conclusions can be drawn for the product $K(\mathrm{j}w)G(\mathrm{j}w)$ concerning perturbations in the plant input. The only difference is that, now, v denotes the degree of normality associated with the optimal condition number of the eigenvector matrix of $K(\mathrm{j}w)G(\mathrm{j}w)$.

Conclusions (1)–(4) ensure that only when $G(\mathrm{j}w)$ is not normal at a frequency band containing the crossover frequency ω_{c}, a static normalizing pre-compensator K_{p} should be designed in order to make $G(\mathrm{j}w)K_{\mathrm{p}}$ approximately normal in this frequency band.

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Therefore, instead of designing a dynamic normalizing pre-compensator, as proposed in [16, 19], it suffices to design a static pre-compensator that normalizes the plant at the vicinity of ω_c.

It can be easily proved that although the resulting controller

$$K(s) = K_p K_c(s) \tag{11}$$

where K_p is the normalizing pre-compensator and $K_c(s)$ is a controller that commutes with $G(s)K_p$ can be designed to improve the closed-loop system robustness with respect to perturbations in the plant output, it may be possible that the closed-loop system robustness with respect to perturbations in plant input be poor. This difficulty can be overcome by introducing another objective on the design of K_p, namely that K_p should make both $K_p K_c(jw)G(jw)$ and $G(jw)K_p K_c(jw)$ approximately normal at the frequency range of interest.

3. NORMALIZING PRE-COMPENSATORS

3.1. Problem formulation

With the view to considering the normalization of both $G(s)K(s)$ and $K(s)G(s)$, the so-called reversed-frame-normalizing-controllers has been introduced [20]. It can be seen that both $G(s)K(s)$ and $K(s)G(s)$ are normal if and only if the singular-vector frames of $K(s)$ are those of $G(s)$ taken in reversed order, as stated in the following lemma.

Lemma 3

Suppose that $G, K \in \mathbb{C}^{m\times m}$ are both of rank m and let

$$G = Y\Sigma U^{\star} \tag{12}$$

be a singular value decomposition of G, where $\Sigma = \text{diag}\{\sigma_i, i = 1, \ldots, m\}$. Then GK and KG are both normal if and only if

$$K = U\Gamma_K Y^{\star} \tag{13}$$

for some nonsingular diagonal matrix $\Gamma_K \in \mathbb{C}^{m\times m}$.

Proof

See [20]. □

According to Lemma 3, the characteristic loci of $G(s)K(s)$ are at their least sensitivity to perturbations in the plant input and output if and only if the controller $K(s)$ has the structure given by Equation (13). However, in this paper, the controller is defined by Equation (11), which implies that K_p must have a specific structure so that $G(s)K(s)$ and $K(s)G(s)$ be both normal, as stated below.

Theorem 3

Suppose that the product $GK \in \mathbb{C}^{m\times m}$ is of rank m and has all eigenvalues distinct. In addition, assume that all eigenvalues of GK_p are also distinct and let $K = K_p K_c$, where K_c has the same eigenvector matrix as GK_p. Then, GK and KG are both normal matrices if and only if

$$K_p = U\Phi Y^{\star} \tag{14}$$

where Φ is a nonsingular diagonal matrix.

Proof

($\Rightarrow$) If GK and KG are normal matrices, then, according to Lemma 3,

$$K = K_p K_c = U\Gamma_K Y^{\star} \tag{15}$$

Therefore, from Equations (12) and (13), it can be seen that $GK = Y\Sigma U^{\star} U\Gamma_K Y^{\star} = Y\Sigma\Gamma_K Y^{\star}$, which is a spectral decomposition of GK. Since GK_p is assumed to have distinct eigenvalues and K_c and GK_p share the same eigenvector matrices, then $K_c = Y\Lambda_c Y^{\star}$, where Λ_c is the eigenvalue matrix of K_c, is a spectral decomposition of K_c. Using Equation (15), we obtain

$$K_p = U\Gamma_K \Lambda_c^{-1} Y^{\star} \tag{16}$$

which is equivalent to Equation (14) for $\Phi = \Gamma_K \Lambda_c^{-1}$.
($\Leftarrow$) The proof is straightforward and will be omitted. □

Assuming that, for a given frequency w_0, $K_c(jw_0)$ is chosen in such a way that all eigenvalues of $G(jw_0)K(jw_0)$ are distinct, then exact normality of $G(jw_0)K(jw_0)$ and $K(jw_0)G(jw_0)$ are achieved through the choice of a nonsingular diagonal matrix Φ such that $G(jw_0)K_p$ has distinct eigenvalues and by computing K_p in accordance with Equation (14).

However, in this paper, the computation of a normalizing static pre-compensator is considered, which implies that $K_{\rm p}$ must be a real matrix. Since the matrices $U(\mathrm{j}w_0), Y(\mathrm{j}w_0) \in \mathbb{C}^{m\times m}$, then, only for special cases it is possible to obtain $K_{\rm p}$ real; in [20] the design of static pre-compensators with the structure given by Equation (14) is considered in two cases where it is possible to obtain real pre-compensators, namely that, at d.c. frequency ($\omega_0 = 0$) and at very high frequencies ($\omega_0 \to \infty$). In these cases $U(\mathrm{j}w_0), Y(\mathrm{j}w_0) \in \mathbb{R}^{m\times m}$. However, the need for normalization is more critical at frequencies, where the characteristic loci are close to the critical point $-1+\mathrm{j}0$, i.e. at intermediate frequencies, as pointed out in Section 2.

Since exact normalities of $G(\mathrm{j}w)K$ and $KG(\mathrm{j}w)$ at intermediate frequencies by means of a complex matrix $K = K_{\rm p}K_{\rm c}$ ($K_{\rm p} \in \mathbb{R}^{m\times m}$) cannot, in general, be simultaneously obtained, it is more realistic to seek approximate normality. The following results relates the approximate normality of $G(\mathrm{j}w)K(\mathrm{j}w)$ and $K(\mathrm{j}w)G(\mathrm{j}w)$ to the approximate normality of $G(\mathrm{j}w)K_{\rm p}$ and $K_{\rm p}G(\mathrm{j}w)$, respectively.

Theorem 4
Let $K(s)$ be given by Equation (11), and assume that G and K denote, respectively, the values of $G(\mathrm{j}w_0)$ and $K(\mathrm{j}w_0)$ at a given frequency $\omega_0 \in \mathbb{R}$. In addition, assume that all eigenvalues of $GK_{\rm p}$ are distinct and $K_{\rm c}$ has the same eigenvector matrix as $GK_{\rm p}$. Then (i) GK is approximately normal if $GK_{\rm p}$ is approximately normal and (ii) assuming that $K_{\rm p}$ is nonsingular, KG is approximately normal if $K_{\rm p}G$ is approximately normal.

Proof

(i) The proof is straightforward and comes from the fact that since $K_{\rm c}$ has the same eigenvectors as $GK_{\rm p}$, then $GK_{\rm p}$ and GK have the same eigenvector matrices.

(ii) Let a singular value decomposition of G be given by Equation (12) and for a nonsingular matrix $K_{\rm p}$, define

$$M(\mathrm{j}w_0) = U(\mathrm{j}w_0)^\star K_{\rm p} Y(\mathrm{j}w_0) \tag{17}$$

It follows from Equations (12) and (17) that:

$$GK_{\rm p} = Y\Sigma U^\star U M Y^\star = Y\Sigma M Y^\star \tag{18}$$

Suppose now that $\Sigma M = W_{\Sigma M}\Lambda_{\Sigma M}V_{\Sigma M}$, where $\Lambda_{\Sigma M}$ is a diagonal matrix, $W_{\Sigma M}$ is an eigenvector matrix of ΣM and $V_{\Sigma M} = W_{\Sigma M}^{-1}$. Then, $GK_{\rm p}$ can be written as:

$$GK_{\rm p} = Y W_{\Sigma M}\Lambda_{\Sigma M}V_{\Sigma M}Y^\star \tag{19}$$

Since, by assumption, $K_{\rm c}$ has the same eigenvector matrix as $GK_{\rm p}$, a spectral decomposition of $K_{\rm c}$ can always be given as

$$K_{\rm c} = Y W_{\Sigma M}\Lambda_{\rm c}V_{\Sigma M}Y^\star \tag{20}$$

where $\Lambda_{\rm c}$ is a diagonal matrix formed with the eigenvalues of $K_{\rm c}$. Thus, according to Equations (17), (20) and (12), and after simple algebraic manipulations, KG can be written as:

$$KG = U M W_{\Sigma M}\Lambda_{\rm c}\Lambda_{\Sigma M}V_{\Sigma M}M^{-1}U^\star \tag{21}$$

Therefore, for any commutative controller $K_{\rm c}$, an eigenvector matrix of KG will be given by $UMW_{\Sigma M}$. In particular, for $K_{\rm c} = I$, $K_{\rm p}G$ will have the following spectral decomposition:

$$K_{\rm p}G = U M W_{\Sigma M}\Lambda_{\Sigma M}V_{\Sigma M}M^{-1}U^\star \tag{22}$$

Since, by assumption, all eigenvalues of $K_{\rm p}G$ are distinct, then an eigenvector matrix of $K_{\rm p}G$ is equal to $UMW_{\Sigma M}$ up to a scaling to have the smallest condition number according to Equation (2). Therefore, if the eigenvector matrix of $K_{\rm p}G$ has optimal condition number approximately equal to one, i.e. if $K_{\rm p}G$ is approximately normal, then KG is also approximately normal. □

Theorems 3 and 4 lead to the formulation of the following optimization problem to find a real pre-compensator $K_{\rm p}$ that approximately normalizes $G(\mathrm{j}w)K_{\rm p}$ and $K_{\rm p}G(\mathrm{j}w)$ at $\omega = \omega_0$

$$\min_{(K_{\rm p},\Phi)} \|K_{\rm p} - U\Phi Y^\star\|_{\mathscr{F}}^2 \tag{23}$$

where $U = U(\mathrm{j}w_0)$ and $Y = Y(\mathrm{j}w_0)$ are obtained according to Equation (12) from a singular value decomposition of $G(\mathrm{j}\omega_0)$ and $\|\cdot\|_{\mathscr{F}}$ denotes the Frobenius norm, subject to the following constraints: $K_{\rm p} \in \mathbb{R}^{m\times m}$ and nonsingular, and $\Phi \in \mathbb{C}^{m\times m}$, diagonal and nonidentically zero.

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Optimization problem (23) leads to a pre-compensator that approximately normalizes $G(\mathrm{j}w)K_\mathrm{p}$ and $K_\mathrm{p}G(\mathrm{j}w)$, as shown by the following results.

Lemma 4 (*Bauer and Fike* [*24*])
Let $A\in\mathbb{C}^{m\times m}$ be a diagonalizable matrix and $A=W\Lambda V$, where $V=W^{-1}$, be a spectral decomposition of A and let $\Delta\in\mathbb{C}^{m\times m}$. In addition, assume that $\hat{\lambda}$ is an eigenvalue of $A+\Delta$. Then there exists an eigenvalue λ_i of A such that

$$|\hat{\lambda}-\lambda_i|\leqslant\bar{\sigma}(W)\bar{\sigma}(V)\bar{\sigma}(\Delta)=\mathscr{C}(W)\bar{\sigma}(\Delta) \tag{24}$$

Proof
See [21]. □

Lemma 5
Let $P\in\mathbb{C}^{m\times m}$ be such that $P=\mathrm{e}^{\mathrm{j}\Psi}+P_0$, where Ψ is a diagonal matrix, $P_0\in\mathbb{C}^{m\times m}$ and $\bar{\sigma}(P_0)=\varepsilon<1$. Then,

$$\mathscr{C}(P)\leqslant\frac{1+\varepsilon}{1-\varepsilon} \tag{25}$$

Proof
The proof can be easily obtained using properties of the singular values. □

Theorem 5
Let $G=Y\Sigma U^\star$ be a singular value decomposition of G, $\Phi\in\mathbb{C}^{m\times m}$ be a diagonal matrix and $K_\mathrm{p}\in\mathbb{R}^{m\times m}$. Define

$$E=U^\star K_\mathrm{p}Y-\Phi \tag{26}$$

and $Z=\Sigma\Phi=\mathrm{diag}\{z_1,z_2,\ldots,z_m\}$. Then, if $z_i\neq z_\ell$, $i\neq\ell$, $i,\ell\in\mathscr{I}_m$, where $\mathscr{I}_m=\{1,2,\ldots,m\}$ and

$$\frac{2\sqrt{m}\|\Sigma\|_\mathscr{F}}{\min_{i,\ell\in\mathscr{I}_m,i\neq\ell}|z_i-z_\ell|}\|E\|_\mathscr{F}\rightarrow 0 \tag{27}$$

then $v_{GK_\mathrm{p}}\rightarrow 0$ and $v_{K_\mathrm{p}G}\rightarrow 0$, where v_{GK_p} and $v_{K_\mathrm{p}G}$ denote the deviation from normality of GK_p and $K_\mathrm{p}G$, respectively, given according to Definition 4.

Proof
Let

$$G_\mathrm{p}=GK_\mathrm{p}=W_\mathrm{p}\Lambda_\mathrm{p}V_\mathrm{p} \tag{28}$$

be a spectral decomposition of GK_p where $V_\mathrm{p}=W_\mathrm{p}^{-1}$. Since Y is a unitary matrix, then any eigenvector matrix of G_p formed with eigenvectors with unity Euclidean norm can be written as

$$W_\mathrm{p}=YP \tag{29}$$

where $P\in\mathbb{C}^{m\times m}$ is a nonsingular matrix with all columns with Euclidean norm equal to one. Thus, using Equations (28), (29) and (26), it can be verified that:

$$\Sigma\Phi+\Sigma E=P\Lambda_\mathrm{p}P^{-1} \tag{30}$$

According to Lemma 4, associated with each diagonal entry of $\Sigma\Phi$, $\sigma_i\phi_i$, there exists an eigenvalue of Λ_p, λ_{p_i}, such that

$$|\sigma_i\phi_i-\lambda_{p_i}|\leqslant\bar{\sigma}(\Sigma E)\leqslant\|\Sigma\|_\mathscr{F}\|E\|_\mathscr{F} \tag{31}$$

Therefore, defining the complex matrix

$$\hat{\Delta}=\Sigma\Phi-\Lambda_\mathrm{p} \tag{32}$$

and using inequality (31) and the fact that $\hat{\Delta}$ is a diagonal matrix, it follows that:

$$\bar{\sigma}(\hat{\Delta})=\max_i|\sigma_i\phi_i-\lambda_{p_i}|\leqslant\|\Sigma\|_\mathscr{F}\|E\|_\mathscr{F} \tag{33}$$

Substituting Λ_p given by Equation (32), in Equation (30), and after simple algebraic manipulations, one obtains:

$$P\Sigma\Phi-\Sigma\Phi P=\Sigma EP+P\hat{\Delta} \tag{34}$$

Computing the Frobenius norm of both sides of Equation (34) and using properties of the norm of matrices, it results in:

$$\begin{aligned}&\|P\Sigma\Phi-\Sigma\Phi P\|_\mathscr{F}\\&\quad\leqslant\|\Sigma\|_\mathscr{F}\|E\|_\mathscr{F}\|P\|_\mathscr{F}+\|P\hat{\Delta}\|_\mathscr{F}\end{aligned} \tag{35}$$

Since all columns of P have Euclidean norm equal to one, then $\|P\|_\mathscr{F}=\sqrt{m}$. In addition, using Equation (33), one can write

$$\begin{aligned}\|P\hat{\Delta}\|_\mathscr{F}&=\left(\sum_{i=1}^m|\hat{\delta}_i|^2\|\underline{p}_i\|_2^2\right)^{1/2}\\&\leqslant\sqrt{m}\max_i|\hat{\delta}_i|=\sqrt{m}\bar{\sigma}(\hat{\Delta})\\&\leqslant\sqrt{m}\|\Sigma\|_\mathscr{F}\|E\|_\mathscr{F}\end{aligned} \tag{36}$$

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where $\hat{\delta}_i$ denotes the ith entry of the main diagonal of $\hat{\Delta}$ and $\underline{p}_i$, the ith column of P. Therefore, inequality (35) can be written as:

$$\|P\Sigma\Phi-\Sigma\Phi P\|_{\mathscr{F}}\leqslant 2\sqrt{m}\|\Sigma\|_{\mathscr{F}}\|E\|_{\mathscr{F}} \tag{37}$$

Defining $Z=\Sigma\Phi$, then, it can be seen that the (i,ℓ) entry of $ZP-PZ$ is given by $(z_i-z_\ell)p_{i\ell}$, where $p_{i\ell}$ denotes entry (i,ℓ) of P. Thus,

$$\|ZP-PZ\|_{\mathscr{F}}^2=\sum_{\ell=1}^{m}\sum_{i=1,i\neq\ell}^{m}|z_i-z_\ell|^2|p_{i\ell}|^2 \tag{38}$$

Therefore, the following relationship can be established:

$$\begin{aligned}\left(\sum_{\ell=1}^{m}\sum_{i=1,i\neq\ell}^{m}|p_{i\ell}|^2\right)^{1/2}&\min_{i\neq\ell}|z_i-z_\ell|\\ &\leqslant\|P\Sigma\Phi-\Sigma\Phi P\|_{\mathscr{F}}\\ &\leqslant\sqrt{m}\|E\|_{\mathscr{F}}\|\Sigma\|_{\mathscr{F}}\end{aligned} \tag{39}$$

Assuming now that $\|E\|_{\mathscr{F}}$ satisfies the condition given by (27), then

$$\left(\sum_{\ell=1}^{m}\sum_{i=1,i\neq\ell}^{m}|p_{i\ell}|^2\right)^{1/2}\rightarrow 0 \tag{40}$$

Since all columns of P have Euclidean norm equal to one, inequality (40) implies that P is approximately unitary and diagonal. Therefore, according to Lemma 5, $\mathscr{C}(P)\rightarrow 1$ and since $\mathscr{C}(W_{\mathrm{p}})=\mathscr{C}(YP)=\mathscr{C}(P)\geqslant\mathscr{C}^{\mathrm{opt}}(P)$, then it is straightforward to see that $v_{GK_{\mathrm{p}}}\rightarrow 0$.

In order to show that $v_{K_{\mathrm{p}}G}\rightarrow 0$ when condition (27) is satisfied, it suffices to follow the same steps as the proof above for $K_{\mathrm{p}}G$; the only difference is that, in this case the eigenvector matrix of $K_{\mathrm{p}}G$, $\hat{W}_{\mathrm{p}}$ must satisfy $\hat{W}_{\mathrm{p}}=UP$, where U is the input principal direction matrix of a singular value decomposition of G, and P and $\hat{W}_{\mathrm{p}}$ have all columns with Euclidean norm equal to one. □

Theorem 5 shows that if a pair $(K_{\mathrm{p}}^{\mathrm{opt}},\Phi^{\mathrm{opt}})$, solution to optimization problem (23), satisfies condition (27) then $G(\mathrm{j}w_0)K_{\mathrm{p}}^{\mathrm{opt}}$ and $K_{\mathrm{p}}^{\mathrm{opt}}G(\mathrm{j}w_0)$ will both have eigenvector matrices with optimal condition numbers approximately equal to one, being, therefore, approximately normal.

3.2. Solution to the optimization problem

Optimization problem (23) can be written as

$$\min_{K_{\mathrm{p}}\in\mathbb{R}^{m\times m},\Phi\ \text{nonidentically zero}} J(K_{\mathrm{p}},\Phi) \tag{41}$$

where

$$J(K_{\mathrm{p}},\Phi)=\|K_{\mathrm{p}}-U\Phi Y^{\star}\|_{\mathscr{F}}^2 \tag{42}$$

The solution to this problem is given by the following theorem.

Theorem 6

Let U_R (U_I) and Y_R (Y_I) denote the real (imaginary) parts of U and Y, respectively, for a given frequency ω_0 and $\underline{u}_{R_i}$ ($\underline{u}_{I_i}$), $i=1,2,\ldots,m$ denote the ith column of U_R (U_I) and $y_{R_{ij}}$ ($y_{I_{ij}}$) denote the (i,j) entry of Y_R (Y_I). Define the matrix

$$A=[A_{RR}+A_{II}\ \ A_{RI}-A_{IR}] \tag{43}$$

where A_{RR}, A_{II}, A_{RI} and A_{IR} are defined as

$$A_{\alpha\beta}=\begin{bmatrix}\underline{u}_{\alpha_1}y_{\beta_{11}} & \underline{u}_{\alpha_2}y_{\beta_{12}} & \cdots & \underline{u}_{\alpha_m}y_{\beta_{1m}}\\ \underline{u}_{\alpha_1}y_{\beta_{21}} & \underline{u}_{\alpha_2}y_{\beta_{22}} & \cdots & \underline{u}_{\alpha_m}y_{\beta_{2m}}\\ \vdots & \vdots & & \vdots\\ \underline{u}_{\alpha_1}y_{\beta_{m1}} & \underline{u}_{\alpha_2}y_{\beta_{m2}} & \cdots & \underline{u}_{\alpha_m}y_{\beta_{mm}}\end{bmatrix} \tag{44}$$

with α and β being replaced with R or I where appropriate. In addition, write

$$K_{\mathrm{p}}=[\underline{k}_1\ \ \underline{k}_2\ \ \ldots\ \ \underline{k}_m] \tag{45}$$

where $\underline{k}_i$ denotes the ith column of K_{p} and $\Phi=\Phi_R+j\Phi_I$, where Φ_R and Φ_I are diagonal matrices with real entries. Form the vectors

$$\underline{\phi}=[\underline{\phi}_R^{\mathrm{T}}\ \ \underline{\phi}_I^{\mathrm{T}}]^{\mathrm{T}} \tag{46}$$

$$\underline{k}=[\underline{k}_1^{\mathrm{T}}\ \ \underline{k}_2^{\mathrm{T}}\ \ \ldots\ \ \underline{k}_m^{\mathrm{T}}]^{\mathrm{T}} \tag{47}$$

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where $\underline{\phi}_R$ and $\underline{\phi}_I$ are column vectors formed with the diagonal entries of Φ_R and Φ_I. Let $A = Y_A \Sigma_A U_A^\star$ be a singular value decomposition of A, σ_{A_1} be the largest singular value of A and $\underline{u}_{A_1}$ ($\underline{y}_{A_1}$) the first column of U_A (Y_A). Assuming that $\underline{\phi}$ is normalized to have unit Euclidean norm, then $\min \|K_p - U\Phi Y^\star\|_{\mathscr{F}}^2 = 1 - \sigma_{A_1}^2$, and is achieved by the pair (K_p^{opt}, Φ^{opt}) obtained from the entries of $\underline{\phi}^{opt} = \underline{u}_{A_1}$ and

$$\underline{k}^{opt} = \sigma_{A_1} \underline{y}_{A_1} \tag{48}$$

Proof
After some simple algebraic manipulations, $J(K_p, \Phi)$ can be written as

$$J(K_p, \Phi) = \mathrm{tr}(K_p K_p^T) - 2\,\mathrm{Re}\{\mathrm{tr}(U^\star K_p Y \Phi^\star)\} + \mathrm{tr}(\Phi^\star \Phi) \tag{49}$$

where $\mathrm{Re}\{\cdot\}$ and $\mathrm{tr}(\cdot)$ denote, respectively, the real part and the trace of a complex matrix. Since, $U = U_R + jU_I$, $Y = Y_R + jY_I$ and $\Phi = \Phi_R + j\Phi_I$, then $J(K_p, \Phi)$ can also be written as

$$J(K_p, \Phi) = \mathrm{tr}(K_p K_p^T) - 2H + \mathrm{tr}(\Phi^\star \Phi) \tag{50}$$

where

$$H = \mathrm{tr}(U_R^T K_p Y_R \Phi_R) + \mathrm{tr}(U_R^T K_p Y_I \Phi_I) - \mathrm{tr}(U_I^T K_p Y_R \Phi_I) + \mathrm{tr}(U_I^T K_p Y_I \Phi_R) \tag{51}$$

Notice that each term on the right-hand side of Equation (51) has the same structure, i.e. $\mathrm{tr}(U_\alpha K_p Y_\beta \Phi_\gamma)$, where α, β and γ are replaced with R or I, when appropriate. Defining the vectors $\underline{\phi}$ and $\underline{k}$, according to Equations (46) and (47), then it is straightforward to check that

$$H = \underline{k}^T A \underline{\phi} \tag{52}$$

where A is defined according to Equation (43). Therefore, the problem of minimizing $J(K_p, \Phi)$ given by Equation (50) is equivalent to the problem of minimizing

$$J(\underline{k}, \underline{\phi}) = \underline{k}^T \underline{k} - 2\underline{k}^T A \underline{\phi} + \underline{\phi}^T \underline{\phi} \tag{53}$$

Note that if one fixes the values of the elements of vector $\underline{\phi}$, then the cost function $J(\underline{k}, \underline{\phi})$ describes a paraboloid whose minimum value is given by

$$J_{\min}(\underline{\phi}) = \underline{\phi}^T \underline{\phi} - \underline{k}^T \underline{k} \tag{54}$$

where $\underline{k} = A\underline{\phi}$. Therefore, the problem of minimizing $J(\underline{k}, \underline{\phi})$, given by Equation (53), is equivalent to the following optimization problem:

$$\min_{\underline{\phi}} \underline{\phi}^T (I - A^T A) \underline{\phi} \tag{55}$$

for all $\underline{\phi}$ such that Φ is nonidentically zero. Note that since the cost function of problem (55) is equivalent to the norm defined in Equation (42), then the matrix $(I - A^T A)$ is either positive definite or positive semi-definite. Furthermore, since the Euclidean norm of vector $\underline{\phi}$ is assumed to be equal to one, it can be seen that problem (55) is equivalent to:

$$\max_{\underline{\phi}} \underline{\phi}^T (A^T A) \underline{\phi} \tag{56}$$

Let $A = Y_A \Sigma_A U_A^\star$ be a singular value decomposition of A. Then, the solution to optimization problem (56) is achieved for $\underline{\phi}^{opt} = \underline{u}_{A_1}$ (the first column of U_A), and the maximum value of cost function (56) is given by $\sigma_{A_1}^2$ (the square of the largest singular value of A). Furthermore, it is not hard to check that $\underline{k}^{opt} = A\underline{\phi}^{opt} = \sigma_{A_1} \underline{y}_{A_1}$, where $\underline{y}_{A_1}$ denotes the first column of Y_A. □

Remark 2
In general, choosing $\underline{k} = \sigma_{A_1} \underline{y}_{A_1}$ leads to a nonsingular K_p; this has actually happened in all examples considered so far [25]. However, if it happens that the minimum is achieved with a singular K_p, then a nonsingular K_p must be obtained among the suboptimal solutions of problem (56). In order to do so in a systematic manner, note that any vector $\underline{\phi}$ can always be written as a linear combination of the columns of the matrix U_A, as follows:

$$\underline{\phi} = \alpha_1 \underline{u}_{A_1} + \alpha_2 \underline{u}_{A_2} + \cdots + \alpha_{2m} \underline{u}_{A_{2m}} \tag{57}$$

where it is straightforward to see that $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \cdots = \alpha_{2m} = 0$ for $\underline{\phi}^{opt}$. Therefore, the cost of problem (56) can be written in terms of α_i, $i = 1, 2, \ldots, 2m$, as

$$\underline{\phi}^T (A^T A) \underline{\phi} = \alpha_1^2 \sigma_{A_1}^2 + \alpha_2^2 \sigma_{A_2}^2 + \cdots + \alpha_{2m}^2 \sigma_{A_{2m}}^2 \tag{58}$$

with the constraint

$$\alpha_1^2+\alpha_2^2+\cdots+\alpha_{2m}^2=1 \tag{59}$$

which is implied by $\|\underline{\phi}\|_2^2=1$. Moreover, since $\underline{k}=A\underline{\phi}$, then using Equation (57), we may write

$$\underline{k}=\alpha_1\sigma_{A_1}\underline{y}_{A_1}+\alpha_2\sigma_{A_2}\underline{y}_{A_2}+\cdots+\alpha_p\sigma_{A_p}\underline{y}_{A_p} \tag{60}$$

where $\underline{y}_{A_i}$, $i=1,\ldots,p$, are the first p columns of the output principal direction matrix Y_A associated with nonzero singular values. Using Equation (60), it is possible to express K_p as

$$K_\mathrm{p}=\alpha_1\sigma_{A_1}Y_{A_1}+\alpha_2\sigma_{A_2}Y_{A_2}+\cdots+\alpha_p\sigma_{A_p}Y_{A_p} \tag{61}$$

where Y_{A_i} is an $m\times m$ matrix formed from $\underline{y}_{A_i}$, $i=1,2,\ldots,p$. Let σ^2 denote the value of cost (58). Therefore, it can be easily checked that

$$\frac{\alpha_1^2}{a_1^2}+\frac{\alpha_2^2}{a_2^2}+\cdots+\frac{\alpha_p^2}{a_\mathrm{p}^2}=1 \tag{62}$$

where $a_i=\sigma/\sigma_{A_i}$, $i=1,2,\ldots,p$. Note that Equations (59) and (62) define, respectively, a hypersphere of radius 1 and a hyper-ellipsoid of semi-axes a_i, $1=1,2,\ldots,p$. Furthermore, the following can be stated about their interception points (it is assumed here, for simplicity, that A has distinct singular values): (i) when $\sigma=\sigma_{A_1}$, then $a_1=1$ and $a_i>1$, $i=2,3,\ldots,p$, the hypersphere is inside the hyper-ellipsoid and is tangent to it at point $(1,0,\ldots,0)$; (ii) as σ decreases, a_1 becomes smaller than 1, which creates several interception points; (iii) when $\sigma=\sigma_\mathrm{p}$, the hyper-ellipsoid becomes internal and tangent to the hypersphere and the unique interception point is $(0,0,\ldots,1)$. Therefore, a nonsingular K_p that leads to a suboptimal solution to optimization problem (56) can be found using a search algorithm.

It may happen that the solution to optimization problem (55) for just one frequency leads to a pre-compensated plant that is approximately normal only at a small vicinity of the frequency ω_0 considered in the optimization problem. If a larger band is required, then it is necessary to consider more than one frequency in the computation of the pre-compensator K_p. This problem is addressed in the next section.

3.3. *Extension to multi-frequencies*

Assume that l distinct frequencies are chosen in the intermediate frequency range. Then, the cost function (42) can be modified in order to compute a static pre-compensator, as follows:

$$J(K_\mathrm{p},\Phi_n)=\sum_{n=1}^{l}\tilde{w}_n\|K_\mathrm{p}-U_n\Phi_nY_n^\star\|_{\mathscr{F}}^2 \tag{63}$$

where $\{\tilde{w}_n\in\mathbb{R}_+^*, n=1,\ldots,l\}$ is a set of weights, $K_\mathrm{p}\in\mathbb{R}^{m\times m}$, $\Phi_n\in\mathbb{C}^{m\times m}$ is a diagonal matrix, and U_n, Y_n are, respectively, the input and output principal direction matrices of a singular value decomposition of $G(\mathrm{j}w_n)$ for $n=1,\ldots,l$. It is not hard to check that if $K_\mathrm{p}^{\mathrm{opt}}$ and Φ_n^{opt}, $n=1,2,\ldots,l$, satisfy condition (27) then $G(\mathrm{j}w_n)K_\mathrm{p}^{\mathrm{opt}}$ and $K_\mathrm{p}^{\mathrm{opt}}G(\mathrm{j}w_n)$ will be approximately normal at the selected frequencies ω_n.

The problem of minimizing $J(K_\mathrm{p},\Phi_n)$ can be converted to an eigenvalue problem, whose solution $(K_\mathrm{p}^{\mathrm{opt}},\Phi_1^{\mathrm{opt}},\Phi_2^{\mathrm{opt}},\ldots,\Phi_l^{\mathrm{opt}})$ is obtained according to the following Theorem.

Theorem 7

Let U_{n_R} (U_{n_I}) and Y_{n_R} (Y_{n_I}) denote the real (imaginary) parts of U_n and Y_n, respectively, for a given frequency ω_n and $\underline{u}_{n_{R_i}}$ $(\underline{u}_{n_{I_i}})$, $i=1,2,\ldots,m$, denote the ith column of U_{n_R} (U_{n_I}) and $y_{n_{R_{ij}}}$ $(y_{n_{I_{ij}}})$ denote the (i,j) entry of Y_{n_R} (Y_{n_I}). Define the matrix

$$A_n=[A_{n_{RR}}+A_{n_{II}}\quad A_{n_{RI}}-A_{n_{IR}}]\qquad n=1,2,\ldots,l \tag{64}$$

where $A_{n_{RR}}$, $A_{n_{II}}$, $A_{n_{RI}}$ and $A_{n_{IR}}$ are defined as

$$A_{n_{\alpha\beta}}=\begin{bmatrix}\underline{u}_{n_{\alpha_1}}y_{n_{\beta_{11}}} & \underline{u}_{n_{\alpha_2}}y_{n_{\beta_{12}}} & \cdots & \underline{u}_{n_{\alpha_m}}y_{n_{\beta_{1m}}}\\ \underline{u}_{n_{\alpha_1}}y_{n_{\beta_{21}}} & \underline{u}_{n_{\alpha_2}}y_{n_{\beta_{22}}} & \cdots & \underline{u}_{n_{\alpha_m}}y_{n_{\beta_{2m}}}\\ \vdots & \vdots & & \vdots\\ \underline{u}_{n_{\alpha_1}}y_{n_{\beta_{m1}}} & \underline{u}_{n_{\alpha_2}}y_{n_{\beta_{m2}}} & \cdots & \underline{u}_{n_{\alpha_m}}y_{n_{\beta_{mm}}}\end{bmatrix} \tag{65}$$

with α and β being replaced with R or I, where appropriate, and form the following matrix:

$$\tilde{A}=[A_1\quad A_2\quad\ldots\quad A_l] \tag{66}$$

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In addition, form the vector

$$\underline{\tilde{\phi}} = [\tilde{w}_1 \underline{\phi}_1^{\mathrm{T}} \;\; \tilde{w}_2 \underline{\phi}_2^{\mathrm{T}} \;\; \ldots \;\; \tilde{w}_l \underline{\phi}_l^{\mathrm{T}}]^{\mathrm{T}} \tag{67}$$

where

$$\underline{\phi}_n = [\underline{\phi}_{n_R}^{\mathrm{T}} \;\; \underline{\phi}_{n_I}^{\mathrm{T}}]^{\mathrm{T}}, \quad n = 1, 2, \ldots, l \tag{68}$$

with $\underline{\phi}_{n_R}$ and $\underline{\phi}_{n_I}$ being column vectors formed with the diagonal entries of Φ_{n_R} and Φ_{n_I} (Φ_{n_R} and Φ_{n_I} are diagonal matrices corresponding to the real and imaginary parts of Φ_n). Write

$$K_{\mathrm{p}} = [\underline{k}_1 \;\; \underline{k}_2 \;\; \ldots \;\; \underline{k}_m] \tag{69}$$

and stack its columns to form the following vector:

$$\underline{k} = [\underline{k}_1^{\mathrm{T}} \;\; \underline{k}_2^{\mathrm{T}} \;\; \ldots \;\; \underline{k}_m^{\mathrm{T}}]^{\mathrm{T}} \tag{70}$$

Finally, let λ_{p_1} and $\underline{v}_{p_1}$ denote the smallest eigenvalue and the corresponding eigenvector of the matrix

$$P = \mathrm{diag}\left\{ \frac{1}{\tilde{w}_1} I_{2m}, \frac{1}{\tilde{w}_2} I_{2m}, \ldots, \frac{1}{\tilde{w}_l} I_{2m} \right\} - \frac{1}{\sum_{n=1}^{l} \tilde{w}_n} \tilde{A}^{\mathrm{T}} \tilde{A} \tag{71}$$

where I_{2m} denote the identity matrix of order $2m$. Assuming that $\underline{\phi}$ is normalized to have unit Euclidean norm, then $\min \sum_{n=1}^{l} \tilde{w}_n \| K_{\mathrm{p}} - U_n \Phi_n Y_n^{\star} \|_{\mathscr{F}}^2 = \lambda_{p_1}$ and is achieved by the $(l+1)$-tuple $(K_{\mathrm{p}}^{\mathrm{opt}}, \Phi_1^{\mathrm{opt}}, \Phi_2^{\mathrm{opt}}, \ldots, \Phi_l^{\mathrm{opt}})$ obtained from the entries of $\underline{\tilde{\phi}}^{\mathrm{opt}} = \underline{v}_{p_1}$ and

$$\underline{k}^{\mathrm{opt}} = \frac{1}{\sum_{n=1}^{l} \tilde{w}_n} \tilde{A} \underline{v}_{p_1} \tag{72}$$

Proof
It is not hard to check that Equation (63) can be rewritten as

$$J(\underline{k}, \underline{\phi}_n) = \sum_{n=1}^{l} \tilde{w}_n (\underline{k}^{\mathrm{T}} \underline{k} - 2 \underline{k}^{\mathrm{T}} A_n \underline{\phi}_n + \underline{\phi}_n^{\mathrm{T}} \underline{\phi}_n) \tag{73}$$

where A_n and ϕ_n are defined according to Equations (64) and (68), respectively. Using the definitions of $\tilde{A}$ and $\underline{\tilde{\phi}}$, given by Equations (66) and (67), respectively, then Equation (73) becomes:

$$J(\underline{k}, \underline{\phi}_n) = \left(\sum_{n=1}^{l} \tilde{w}_n \underline{k}^{\mathrm{T}} \underline{k} \right) - 2 \underline{k}^{\mathrm{T}} [A_1 \;\; A_2 \;\; \ldots \;\; A_l] \times \begin{bmatrix} \tilde{w}_1 \underline{\phi}_1 \\ \tilde{w}_2 \underline{\phi}_2 \\ \vdots \\ \tilde{w}_l \underline{\phi}_l \end{bmatrix} + \tilde{w}_1 \underline{\phi}_1^{\mathrm{T}} \underline{\phi}_1 + \tilde{w}_2 \underline{\phi}_2^{\mathrm{T}} \underline{\phi}_2 + \cdots + \tilde{w}_l \underline{\phi}_l^{\mathrm{T}} \underline{\phi}_l \tag{74}$$

Fixing the value of $\underline{\tilde{\phi}}$, the minimum of $J(\underline{k}, \underline{\phi}_n)$ is achieved for

$$\underline{k} = \frac{1}{\sum_{n=1}^{l} \tilde{w}_n} \tilde{A} \underline{\tilde{\phi}} \tag{75}$$

Therefore, substituting $\underline{k}$, given by Equation (75), in Equation (73), and following the same steps as in the proof of Theorem 6, one obtains the following optimization problem:

$$\min_{\underline{\tilde{\phi}}} \underline{\tilde{\phi}}^{\mathrm{T}} P \underline{\tilde{\phi}} \tag{76}$$

where P is defined in accordance with Equation (71). Note that, since minimization problems (76) and (63) are equivalent, then the matrix P is either a positive definite or a semi-definite matrix. Thus, assuming that $\underline{\tilde{\phi}}$ has Euclidean-norm equal to one, then the minimum of Equation (76) is achieved when $\underline{\tilde{\phi}} = \underline{v}_{p_1}$, where $\underline{v}_{p_1}$ denotes the eigenvector associated with the smallest eigenvalue, λ_{p_1}, of P. Moreover, the minimum is given by λ_{p_1}. □

4. NUMERICAL EXAMPLES

In this section, two examples will be used to illustrate the pre-compensator design proposed in this paper: the first example illustrates the improvement that can be

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achieved by considering more than one frequency in the optimization problem; the second example addresses the design of both the normalizing pre-compensator and the commutative controller.

4.1. *Example 1*

Consider the transfer matrix of a linearized model of the vertical plane dynamics of an aircraft [26] given by

$$G(s)=\frac{1}{d(s)}N(s)$$

where $d(s)=s^5+1.5953s^4+1.7572s^3+0.1112s^2+0.0561s$, $N(s)=[n_{ij}(s)]$, and

$$n_{11}(s)=-1.5750s^3-1.1190s^2+1.5409s-0.0816$$
$$n_{12}(s)=0.2909s^2+0.2527s+0.3712$$
$$n_{13}(s)=0.0732s^3-0.0646s^2-1.2125s-0.0204$$
$$n_{21}(s)=-0.12s^4-0.0739s^3-0.5319s^2-0.2458s$$
$$n_{22}(s)=s^4+1.5415s^3+1.6537s^2$$
$$n_{23}(s)=-0.0052s^3+0.1570s^2+0.1828s$$
$$n_{31}(s)=4.419s^3+1.6674s^2+0.1339s$$
$$n_{32}(s)=0.0485s^2+0.3279s$$
$$n_{33}(s)=-1.6650s^3-1.1574s^2-0.0918s$$

Let $\omega_c=10\,\text{rad/s}$ [26] be the desired crossover frequency. As can be seen from Figure 2(a) (solid line), the condition number of the eigenvector matrix of $G(\text{j}w)$ is large for almost all frequencies and approximately equal to seven for $G(\text{j}10)$, which shows that $G(s)$ is not normal at the vicinity of ω_c. Therefore, it is necessary to compute a normalizing pre-compensator in order to normalise both GK_p and K_pG at the vicinity of ω_c. This can be done as follows: (i) form matrix A, according to Equation (43), and compute its singular value decomposition; (ii) compute the largest singular value of A and its corresponding output principal direction, $\underline{y}_{A_1}$; (iii) obtain $\underline{k}^{\text{opt}}$ in accordance with Equation (48) and form K_p using Equations (45) and (47). Proceeding in this way, we obtain the following normalizing pre-compensator:

$$K_{p_1}=\begin{bmatrix}0.3171 & 0.0030 & 0.0731\\ 0.0385 & -0.0138 & 0.0080\\ 0.8953 & 0.0027 & 0.3010\end{bmatrix} \qquad (77)$$

for which, the minimum value of the cost function (55) is equal to 0.000142463442298. From Figure 2(a) (dashed line), it can be seen that $G(\text{j}w)K_{p_1}$ is closer to normal than $G(\text{j}w)$ at almost all frequencies. Thus, according to Theorem 2, the characteristic loci of the open-loop system can be used to give a reliable measure of robust stability as far as uncertainty in the plant output is concerned. However, as shown in Figure 2(a) (dash-dotted line), the optimal condition number of $K_{p_1}G(\text{j}w)$ at $\omega=1\,\text{rad/s}$ is approximately 6.58 and thus the characteristic loci can be sensitive to model uncertainties in the plant input around this frequency. This shows the need for considering the multi-frequency approach of Section 3.3, which is carried out as follows: (i) choose two frequencies $\omega_1=1\,\text{rad/s}$ and $\omega_2=10\,\text{rad/s}$, and, as weights, $\tilde{w}_1=\tilde{w}_2=1$; (ii) form, according to Equation (71), matrix P (a 12×12 dimensional matrix); (iii) compute the smallest eigenvalue of P (λ_{P_1}), and the corresponding eigenvector $\underline{v}_{P_1}$; (iv) obtain $\underline{k}^{\text{opt}}$ according to Equation (72), and K_p according to Equations (70) and (69). Proceeding this way, the following pre-compensator is obtained:

$$K_{p_2}=\begin{bmatrix}0.1851 & -0.0817 & -0.9711\\ -0.0683 & 0.5303 & -0.1237\\ -0.7982 & -0.1022 & 0.0072\end{bmatrix} \qquad (78)$$

for which the value of the cost function (63) is 0.0335. The optimal condition numbers of the eigenvector matrices of $G(\text{j}w)K_{p_2}$ and $K_{p_2}G(\text{j}w)$ are shown in Figure 2(b) (dashed and dash-dotted lines, respectively). It can be seen that, now, both matrices are approximately normal at the frequency range of interest (the optimal condition numbers of the eigenvector

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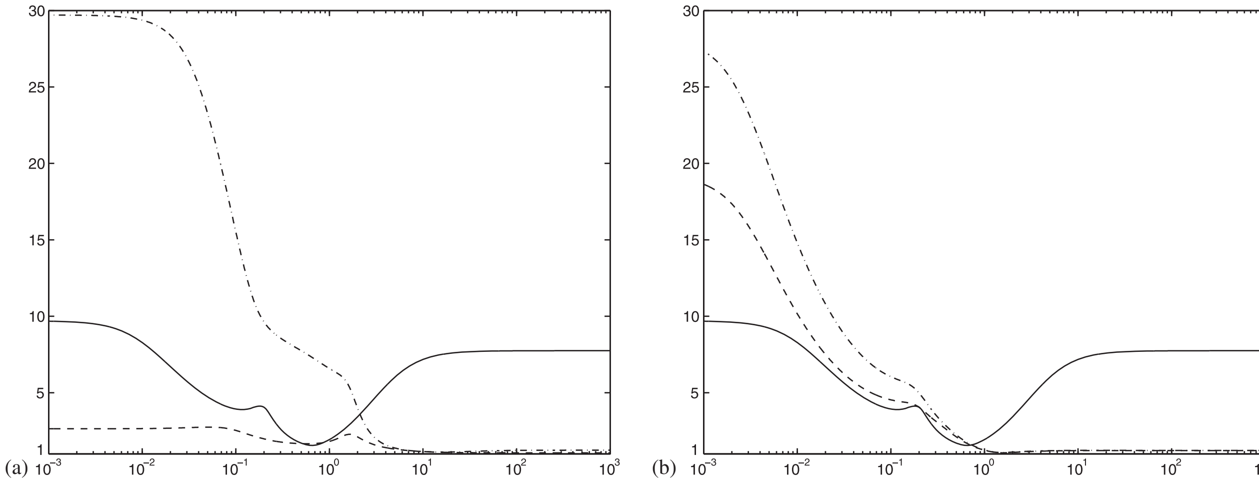

Figure 2. (a) Optimal condition numbers of the eigenvector matrices of $G(\mathrm{j}w)$ (solid line), $G(\mathrm{j}w)K_{p_1}$ (dashed line) and $K_{p_1}G(\mathrm{j}w)$ (dash-dotted line); and (b) optimal condition numbers of the eigenvector matrices of $G(\mathrm{j}w)$ (solid line), $G(\mathrm{j}w)K_{p_2}$ (dashed line) and $K_{p_2}G(\mathrm{j}w)$ (dash-dotted line).

matrices are smaller than 1.5 for all frequencies greater than 0.8 rad/s). Therefore, the CLM can now be effectively applied to the pre-compensated plant assuming uncertainties at both the plant input and output.

4.2. *Example 2*

The plant to be considered here is that proposed in [13], for which the CLM is known to be very sensitive to uncertainties:

$$G(s)=\frac{1}{(s+1)(s+2)}\begin{bmatrix}-47s+2 & 56s\\ -42s & 50s+2\end{bmatrix} \tag{79}$$

It is suggested in [13] that, since the eigenfunctions of $G(s)$ are equal to $\lambda_{g_1}(s)=1/(s+1)$ and $\lambda_{g_2}(s)=2/(s+2)$, then appropriate gain and phase margins can be achieved with the commutative controller $K(s)=I$. However, for this controller, the maximum singular value of the closed-loop transfer matrix $T_{\mathrm{c}}(s)=G(s)K(s)[I+G(s)K(s)]^{-1}$ at $\omega\approx 3\,\mathrm{rad/s}$ is very high (approximately 16.2), showing that the closed-loop system is very sensitive to plant multiplicative uncertainties at the vicinity of this frequency, i.e. even for small variations on $G(s)$, the characteristic loci of the perturbed system differ a great deal from those of the nominal system (the perturbed system may actually become unstable). The low tolerance to plant multiplicative uncertainty can be explained by the lack of normality of $G(s)$ at almost all frequencies: the optimal condition number of the eigenvector matrix of $G(\mathrm{j}w)$ is equal to 196 at all frequencies, except at the dc frequency, when $G(s)$ becomes equal to the identity matrix. It is therefore necessary to design a normalizing pre-compensator for $G(s)$.

Suppose that the desired bandwidth frequency is $\omega_b=1\,\mathrm{rad/s}$, and thus, the characteristic loci should be insensitive to perturbations in the plant input and output in the vicinity of ω_b. Following the same steps as in the first part of Example 1, the following optimal static pre-compensator is obtained:

$$K_{\mathrm{p}}=\begin{bmatrix}0.0216 & -0.7068\\ 0.7068 & 0.0216\end{bmatrix} \tag{80}$$

It can be easily verified that the optimal condition numbers of the eigenvector matrices of $G(\mathrm{j}w)K_{\mathrm{p}}$ and $K_{\mathrm{p}}G(\mathrm{j}w)$, are both equal to one at all frequencies, showing that $G(\mathrm{j}w)K_{\mathrm{p}}$ and $K_{\mathrm{p}}G(\mathrm{j}w)$ are both approximately normal matrices.

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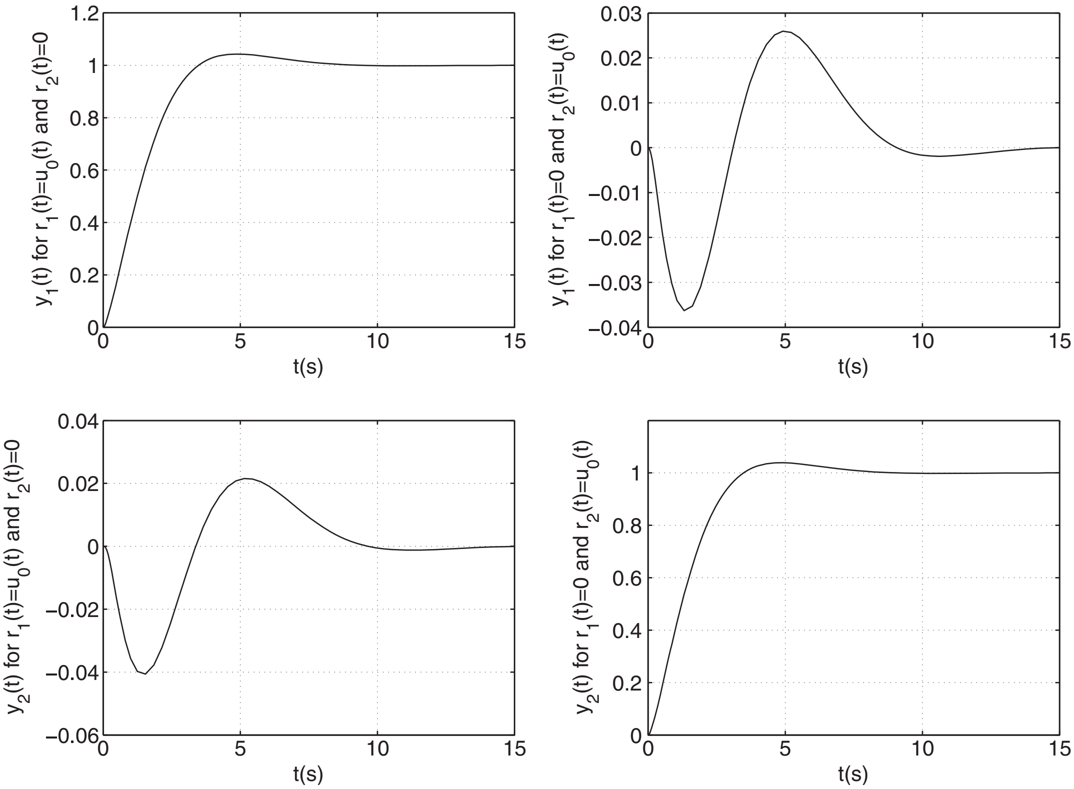

Figure 3. Step responses of the closed-loop system, where $u_0(t)$ is the unit step signal.

Consider now the design of a commutative controller for $G_p(s)=G(s)K_p$ and assume the following design specifications: (S1) rise time smaller than $5s$ for each loop; (S2) low interactions between outputs; (S3) good damping of step responses with peak overshoot no greater than about 10%; (S4) zero steady-state error for step reference input and disturbance; (S5) tolerance to multiplicative uncertainty of size up to $\frac{1}{1.2}$.

Deploying the degrees of freedom available in the parametrization of all rational commutative controllers [18], a controller $K_c(s)$ that commutes with $G_p(s)=G(s)K_p$ can be obtained. The details involved in the calculation of $K_c(s)$ are not presented here since the design of commutative controllers is not the paper main focus; the interested reader is referred to [18]. One of such controllers, has the following transfer function:

$$K_c(s)=\frac{1}{d_{K_c}(s)}\begin{bmatrix}138.4s^4+1790s^3+4210.1s^2+2036.4s+2 & -154.9s^4-1996.8s^3-4627.4s^2-2080.6s+93.4\\ -154.9s^4-2009.7s^3-4792.4s^2-2467.6s-93.4 & 173.5s^4+2243.8s^3+5277.1s^2+2551.5s+2\end{bmatrix}$$

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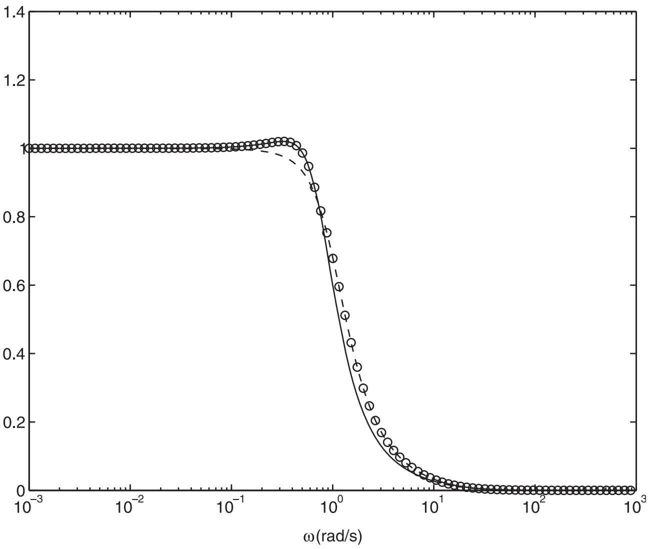

Figure 4. Moduli of eigenfunctions (dashed and solid lines) and maximum singular value (circles) of the closed-loop transfer matrix.

where $d_{K_c}(s) = s^5 + 22.3s^4 + 143.3s^3 + 210.4s^2 + 82.6s$, and has the following properties:

1. It commutes with $G_p(s)$ at all frequencies, as it can be verified by the high degree of commutativity between $G_p(jw)$ and $K_c(jw)$, according to the commutativity indicator

$$e(\omega) = \frac{|\lambda_{Q_i}(jw) - \lambda_{G_{p_i}}(jw)\lambda_{K_{c_i}}(jw)|}{|\lambda_{Q_i}(jw)|} \times 100\%$$

$$i = 1, 2$$

   which is smaller than $2.5 \times 10^{-3}\%$ for $i = 1, 2$, where $Q(s) = G_p(s)K_c(s)$.
2. It satisfies design specifications S1–S4, as one can see from Figure 3.
3. It also satisfies design specification S5, as one can conclude from Figure 4. In this figure, it is shown the moduli of the eigenfunctions (dashed and solid lines) and the maximum singular value (circles) of the closed-loop transfer matrices $T_c(s) = G(s)K(s)[I + G(s)K(s)]^{-1}$; the moduli of the eigenfunctions and the maximum singular value of $T_c'(s) = [I + K(s)G(s)]^{-1}K(s)G(s)$ for the frequency range of interest have not been shown, but it can be easily checked that they coincide with those of the closed-loop transfer function. In addition, note that the largest singular value and the largest modulus of the eigenfunctions are approximately equal for the whole frequency range showing that the pre-compensation scheme proposed here has actually succeeded in providing the basis for the design of a reliable commutative controller, as far as sensitivity to uncertainty in both the plant input and output is concerned.

5. CONCLUSIONS

In this paper, it is shown that, to improve the robustness of a multivariable system compensated with a controller designed in accordance with the CLM, the first step is to pre-compensate the plant transfer matrix with a static pre-compensator that makes the pre-compensated system approximately normal in a frequency band containing the crossover frequency.

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In order to achieve this goal, a systematic methodology for the design of an optimal static normalizing pre-compensator is presented in the paper. The effectiveness of the proposed pre-compensation scheme is illustrated by means of two numerical examples taken from the open literature.

REFERENCES

1. MacFarlane AGJ, Postlethwaite I. The generalized Nyquist stability criterion and multivariable root loci. *International Journal of Control* 1977; **25**(1):81–127. DOI: 10.1080/00207177708922217.
2. MacFarlane AGJ, Belletruti JJ. The characteristic locus design method. *Automatica* 1973; **9**(5):575–588. DOI: 10.1016/0005-1098(73)90043-5.
3. MacFarlane AGJ, Kouvaritakis B. A design technique for linear multivariable systems. *International Journal of Control* 1977; **25**(6):837–874. DOI: 10.1080/00207177708922274.
4. Basilio JC. Multivariable generalised Nyquist design. *Ph.D. Thesis*, University of Oxford, Oxford, U.K., 1995.
5. Mayne DQ. The design of linear multivariable systems. *Automatica* 1973; **9**:201–207.
6. Skogestad S, Morari M. Robust performance of decentralized control systems by independent designs. *Automatica* 1989; **25**(1):119–125. DOI: 10.1016/0005-1098(89)90127-1.
7. Hovd M, Skogestad S. Simple frequency-dependent tools for control system analysis, structure selection and design. *Automatica* 1992; **28**(5):989–996. DOI: 10.1016/0005-1098(92)90152-6.
8. Hovd M, Skogestad S. Sequential design of decentralized controllers. *Automatica* 1994; **30**(10):1601–1607. DOI: 10.1016/0005-1098(94)90099-X.
9. Kosáková A, Veselý V. Independent design of decentralized controllers for specified closed-loop performance. *Proceedings of the European Control Conference*, *ECC'03*, Cambridge, U.K., 2003.
10. Iftar A. Decentralized optimal control with overlapping decompositions. *Proceedings of the 10th IFAC Symposium on Large Scale Systems*, Osaka, Japan, 2004; 605–609.
11. Kosáková A, Veselý V. A frequency domain design technique for robust decentralized controllers. *Preprints of the 16th IFAC World Congress*, Prage, Czech Republic, 2005.
12. Iftar A. Decentralized robust control of large-scale time-delay systems. *Proceedings of the 17th IFAC World Congress*, Seoul, Korea, 2008; 9332–9337.
13. Doyle JC, Stein G. Multivariable feedback design: concepts for a classic/modern synthesis. *IEEE Transactions on Automatic Control* 1981; **26**(1):4–16.
14. Postlethwaite I. Sensitivity of the characteristic gain loci. *Automatica* 1982; **18**(6):709–712. DOI: 10.1016/0005-1098(82)90059-0.
15. Wilkinson JH. *The Algebraic Eigenvalue Problem*. Clarendon Press: Oxford, 1965.
16. Basilio JC, Sahate JA. A normalizing precompensator for the design of effective and reliable commutative controllers. *International Journal of Control* 2000; **73**(14):1280–1297. DOI: 10.1080/002071700421673.
17. Moreira MV, Basilio JC, Kouvaritakis B. Rational stabilizing commutative controllers for unstable plant. *The 44th IEEE Conference on Decision and Control and European Control Conference*, Sevilla, Spain, 2005; 7864–7869.
18. Moreira MV, Basilio JC, Kouvaritakis B. Rational stabilising commutative controllers: parametrisation and characterisation of degrees of freedom. *International Journal of Control* 2006; **79**(12):1601–1612. DOI: 10.1080/00207170600867073.
19. Moreira MV, Basilio JC. Design of normalizing precompensators via alignment of output–input principal directions. *The 44th IEEE Conference on Decision and Control and European Control Conference*, Sevilla, Spain, 2005; 2170–2175.
20. Hung YS, MacFarlane AGJ. *Multivariable Feedback*: *A Quasi-classical Approach*. Springer: Berlin, 1982.
21. Horn RA, Johnson CR. *Matrix Analysis*. Cambridge University Press: Cambridge, 1990.
22. Braatz RD, Morari M. Minimizing the Euclidian condition number. *SIAM Journal on Control and Optimization* 1994; **32**(6):1763–1768. DOI: 10.1137/S0363012992238680.
23. Vidyasagar M. *Control System Synthesis*: *A Factorization Approach*. MIT Press: Cambridge, MA, 1985.
24. Bauer FL, Fike ET. Norms and exclusion theorems. *Numerische Mathematik* 1960; **2**:137–141.
25. Moreira MV. A contribution by means of normalising pre-compensation and parametrisation for the effective use of the characteristic locus method (in Portuguese). *Ph.D. Thesis*, COPPE, Electrical Engineering Program, Federal University of Rio de Janeiro, 2006.
26. Maciejowski JM. *Multivariable Feedback Design*. Addison-wesley Publishing Company: Reading, MA, 1989.

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