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Rational stabilising commutative controllers: parameterisation and characterisation of degrees of freedom

M. V. MOREIRA*†, J. C. BASILIO† and B. KOUVARITAKIS‡

†Universidade Federal do Rio de Janeiro, COPPE – Programa de Engenharia Elétrica, Rio de Janeiro, R.J., Brazil
‡Department of Engineering Science, Oxford University, Parks Road, Oxford, OX2 6UD, UK

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The Characteristic Locus Method constitutes a generalisation of the classical frequency response approach and as such provides a natural platform for design aimed at meeting specifications such as closed-loop stability and dynamic performance. However, to overcome problems of sensitivity to uncertainty, it is necessary to precondition the plant transfer function matrix (TFM) with the view to improving the orthogonality of the eigenvector functions. All that remains then is to use controllers which adjust the frequency response of the eigenfunctions of the TFM while leaving the eigenvectors unaltered. This implies the need for commutative controllers which may be irrational and may not be internally stabilising. The present paper gives a complete characterisation of the class of stabilising rational commutative controllers and derived necessary and sufficient conditions for the existence of this class. These ideas are illustrated by means of case study in which the degrees of freedom contained within the class of commutative controllers are deployed for the meeting design specifications on dynamic performance as well as tolerance to uncertainty.

1. Introduction

The strength of classical frequency response techniques is that they allow an integrated assessment of closed-loop stability and dynamic performance and hence provide a useful basis for the design of feedback loops. Thus critical point encirclements are used to assess stability, whereas dc values provide the necessary information about steady state accuracy. In addition, using the rough but often useful rule of thumb that if $h(t)$ denotes a stable step response, then $h(1/\omega) \approx |H(j\omega)|$ where $H(j\omega)$ denotes the corresponding frequency response, it is possible to relate (clearly only in an approximate manner) system bandwidth to rise time, and maximum M-circle values to peak overshoot. These ideas have a natural extension to multivariable systems through the Characteristic Locus Method (CLM) which for obvious reasons is referred as the Generalized Nyquist approach (MacFarlane and Postlethwaite 1977, MacFarlane and Kouvaritakis 1977, Cameron and Kouvaritakis 1979). The conditions for the assessment of stability and relative stability margins which deploy the Characteristic Loci (the frequency response plot of the eigenfunctions of the appropriate transfer function matrix (TFM)) are necessary and sufficient. This is in contrast to other multivariable frequency response methods which use sufficient only conditions (e.g. based on decoupling or diagonal dominance) and hence could lead to conservative results.

It is known that in the case of badly skewed eigenvectors, eigenvalue/vector decompositions can be sensitive to perturbations (Wilkinson 1965) and therefore in cases like this CLM results may not be robust in the presence of model uncertainty (Doyle and Stein 1981, Moreira and Basilio 2005, Basilio and Sahate 2000). The remedy may be found in the H_∞ methodology, but this does not provide the natural environment for addressing specifications such as speed of response and amount of allowable overshoot. On the other hand, normality (attained through eigenvector orthogonality)

*Corresponding author. Email: moreira@pee.coppe.ufrj.br

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avoids the CLM sensitivity problem and, accordingly, it is possible to design precompensators which improve the degree of normality over particular frequencies of interest (e.g. Basilio and Sahate 2000, Moreira and Basilio 2005). From this point on, CLM design would consist of the derivation of controllers which preserve the eigenvectors (and therefore the degree of normality) while allowing for the adjustment of the CL. This can be achieved through the use of commutative controllers which however may be irrational and therefore difficult to implement and/or may not be internally stabilising (e.g. on account of the introduction of unstable fixed modes). Necessary only conditions for the existence of internally stabilising commutative control have been given (Basilio and Kouvaritakis 1995) and it is the purpose of the current paper to extend these results by deriving a complete characterisation of the class of rational internally stabilising commutative controllers and establishing the necessary and sufficient conditions for the existence of this class. It is demonstrated, through a design study that the degrees of freedom available within the class of commutative controllers can be deployed in meeting a number of objectives that concern dynamic performance as well as afford prescribed margins of robustness; to illustrate the efficacy of the overall proposed strategy, the plant for this study is chosen to be that proposed in Doyle and Stein 1981, for which CLM is known very sensitive to uncertainty.

The layout of the paper is: § 2 presents a brief review of the theory of minimal polynomial bases and fixed modes; the problem of finding stabilising commutative controllers is formulated in § 3; the design study in § 4 demonstrates how the degrees of freedom in the class of commutative controllers can be given up in order to meet a given set of design specifications.

2. Theoretical background

2.1 *Minimal polynomial bases for the right null space of a polynomial matrix*

Let $\mathbb{R}^{p\times q}[s], \mathbb{R}^{p\times q}(s)$ be the sets of $p\times q$ polynomial and rational matrices, respectively. In addition, assume that a matrix $A(s)\in\mathbb{R}^{p\times q}[s]$ ($p<q$ for simplicity) has the following Smith form:

$$\Sigma_A(s)=\begin{bmatrix} \varepsilon_1(s) & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & \varepsilon_2(s) & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varepsilon_p(s) & 0 & \cdots & 0 \end{bmatrix} \quad (1)$$

where $\varepsilon_k(s)\equiv 0,\ k=p-\nu+1,\ldots,p$. In this case the matrix $A(s)$ is said to have a right null space of dimension $\bar{\nu}=q-p+\nu$. This leads to the following definition.

Definition 1: The normal rank of a polynomial matrix $A(s)$ is the integer r such that $\max_{s\in\mathbb{C}}\rho[A(s)]=r$, where $\rho(.)$ denotes the rank of a matrix.

Hence the rank of $A(s)$ can be less than the normal rank, only at a finite number of values of s (i.e. the zeros of the invariant polynomials $\varepsilon_i(s),\ i=1,\ldots,r$). In addition it is always possible to find $\bar{\nu}=q-r$ linearly independent polynomial vectors $\underline{f}(s)$ over the field of rational functions such that $A(s)\underline{f}(s)=\underline{0}$. The latter statement leads to the concept of minimal polynomial basis.

Definition 2: (Forney 1975) Let $F(s)=[\underline{f}_1(s)\ \ \underline{f}_2(s)\ \ldots\ \underline{f}_{\bar{\nu}}(s)]$, where $\deg[\underline{f}_i(s)]=\phi_i$, be a polynomial matrix such that $A(s)F(s)=0$. Then, the columns of $F(s)$ form a minimal polynomial basis for the right null space of $F(s)$ if and only if $\sum_{i=1}^{\bar{\nu}}\phi_i$ is a minimum.

The computation of a minimal polynomial basis for the right null space of a polynomial matrix is straightforward (e.g. one can use the robust algorithm proposed in Basilio and Moreira 2004).

2.2 *Fixed modes of a multivariable system*

Let $G(s)\in\mathbb{R}^{m\times m}(s)$ and form the corresponding characteristic equation of $G(s)$:

$$\Delta(\lambda_g,s)\triangleq\det\big[\lambda_g(s)I_m-G(s)\big]=0, \quad (2)$$

where I_m denotes the identity matrix of order m and $\det[\cdot]$ denotes determinant. In general, $\Delta(\lambda_g,s)$ cannot be expressed as a product of linear factors of λ_g, and thus, the eigenfunctions of a rational matrix are, in general, irrational functions of s. However, $\Delta(\lambda_g,s)$ can be reduced as

$$\Delta(\lambda_g,s)=\Delta_1(\lambda_g,s)\Delta_2(\lambda_g,s)\ldots\Delta_k(\lambda_g,s), \quad (3)$$

where the coefficients $\Delta_i(\lambda_g,s)$ are algebraic functions that are irreducible over the field of rational functions of s. Therefore, each factor $\Delta_i(\lambda_g,s)$ can be written as:

$$b_{i0}(s)\lambda_{g_i}^{\delta_i}+b_{i1}(s)\lambda_{g_i}^{\delta_i-1}+\cdots+b_{i\delta_i}(s)=0, \quad (4)$$

where $b_{ij}(s)$ for $j=0,1,\ldots,\delta_i$, are polynomials in s. The characteristic functions of a square matrix $G(s)\in\mathbb{R}^{m\times m}(s)$ are defined as the set of algebraic functions $\lambda_{g_i}(s)$, for $i=1,\ldots,k$, that satisfies equation (4). Therefore, to any rational matrix $G(s)$, it can be associated a set of algebraic functions $\lambda_{g_i}(s)$ whose values are the eigenfunctions of $G(s)$. This leads to the following definition.

Definition 3: (Poles and zeros of characteristic functions) Let the algebraic equation associated with the characteristic function $\lambda_{g_i}(s)$ be given by equation (4) and also assume that $b_{i0}(s) \neq 0$ and $b_{i\delta_i}(s) \neq 0$. Then, the poles and the zeros of the characteristic function $\lambda_{g_i}(s)$ are the values of $s \in \mathbb{C}$ for which, respectively:

$$b_{i0}(s) = 0, \quad b_{i\delta_i}(s) = 0. \tag{5}$$

Thus the polynomial of the poles and zeros of the characteristic functions of $G(s)$ are defined as

$$p_f(s) = \prod_{i=1}^{k} b_{i0}(s), \; z_f(s) = \prod_{i=1}^{k} b_{i\delta_i}(s). \tag{6}$$

Consider now the Smith–McMillan form of $G(s)$:

$$M_G(s) = \begin{bmatrix} \tilde{M}_G(s)_{r,r} & O_{r,m-r} \\ O_{m-r,r} & O_{m-r,m-r} \end{bmatrix}, \tag{7}$$

where r denotes the normal rank of $G(s)$ and

$$\tilde{M}_G(s) = \operatorname{diag}\begin{bmatrix} \frac{\varepsilon_1(s)}{\sigma_1(s)} & \frac{\varepsilon_2(s)}{\sigma_2(s)} & \cdots & \frac{\varepsilon_r(s)}{\sigma_r(s)} \end{bmatrix}$$

where $\varepsilon_i(s)$, $\sigma_i(s)$, for $i = 1, \ldots, r$, are coprime monic polynomials. Then the following algebraic definition of the poles and zeros of a multivariable TFM (Rosenbrock 1970) is possible.

Definition 4: The finite zeros of $G(s)$ are defined as the set of all zeros of the polynomials $\varepsilon_i(s), i = 1, 2, \ldots, r$. The poles of $G(s)$ are defined as the set of all zeros of $\sigma_i(s), i = 1, 2, \ldots, r$.

Therefore, the pole and zero polynomials of $G(s)$ are given, respectively, by

$$p(s) = \prod_{i=1}^{r} \sigma_i(s), \quad z(s) = \prod_{i=1}^{r} \varepsilon_i(s). \tag{8}$$

It is important to remark (MacFarlane and Postlethwaite 1977; Smith 1981) that not all poles of $G(s)$ are poles of its characteristic functions. Thus

$$p(s) = e(s)p_f(s), \quad z(s) = e(s)z_f(s), \tag{9}$$

where $p_f(s), z_f(s), p(s), z(s)$ are, respectively given by equations (6) and (8), and $e(s)$ is a polynomial whose zeros are the poles of $G(s)$ that are not poles of any of its characteristic functions.

Definition 5: The poles $p_i \in \mathbb{C}$ of $G(s)$ such that $e(p_i) = 0$ are called the fixed modes of $G(s)$.

An important property of the zeros of $e(s)$, that justify the name of fixed modes, is:

Lemma 1: The fixed modes of $G(s)$ are the set of poles of $G(s)$ that remain fixed under any scalar feedback αI_m, where $\alpha \in \mathbb{R}$.

Proof: See Smith (1981). □

3. A parameterisation of all rational stabilising commutative controllers

3.1 *Problem formulation*

Consider the feedback system of figure 1 where $G(s), K(s) \in \mathbb{R}^{m\times m}(s)$ are, respectively, the plant and the controller TFM. In addition let

$$G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s) \tag{10}$$

be a doubly coprime factorisation of $G(s)$ in $RH_\infty^{m\times m}$ (the set of all stable TFM in $\mathbb{R}^{m\times m}(s)$). Thus, there exist $X(s), Y(s), \tilde{X}(s), \tilde{Y}(s) \in RH_\infty^{m\times m}$ which satisfy the Bezout identity

$$\begin{bmatrix} \tilde{X}(s) & -\tilde{Y}(s) \\ \tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) \\ -N(s) & X(s) \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}. \tag{11}$$

Let $K(s)$ be a commutative controller, so that

$$G(s)K(s) = K(s)G(s), \tag{12}$$

and assume that $K(s)$ internally stabilizes the closed-loop system of figure 1, i.e. that belongs to the class (Youla *et al.* 1976, Kucera 1979):

$$\begin{aligned} K(s) &= U(s)V^{-1}(s) = \tilde{V}^{-1}(s)\tilde{U}(s) \\ &= [Y(s) + M(s)Q(s)][X(s) - N(s)Q(s)]^{-1} \\ &= [\tilde{X}(s) - Q(s)\tilde{N}(s)]^{-1}[\tilde{Y}(s) + Q(s)\tilde{M}(s)], \end{aligned} \tag{13}$$

where $Q(s) \in RH_\infty^{m\times m}$, i.e. is rational and has all its poles with negative real part. Then, substitution of $G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$ and $K(s) = U(s)V^{-1}(s) = \tilde{V}^{-1}(s)\tilde{U}(s)$ in equation (12), gives:

$$\begin{aligned} &\tilde{M}^{-1}(s)\tilde{N}(s)U(s)V^{-1}(s) \\ &\quad - \tilde{V}^{-1}(s)\tilde{U}(s)N(s)M^{-1}(s) = O. \end{aligned} \tag{14}$$

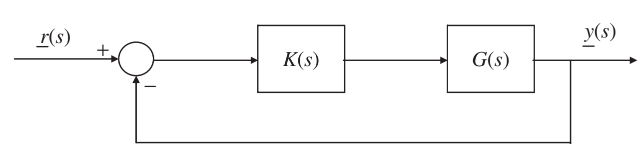

Figure 1. Block diagram of a negative feedback control system

The Bezout identity (equation (11)) is satisfied by $\tilde{V}(s)$, $\tilde{U}(s)$, $V(s)$, $U(s)$ (as well as by $\tilde{X}(s)$, $\tilde{Y}(s)$, $X(s)$, $Y(s)$). Therefore, after some algebraic manipulation, equation (14) reduces to:

$$V(s)\tilde{M}(s) - M(s)\tilde{V}(s) = O. \tag{15}$$

Substituting $V(s) = X(s) - N(s)Q(s)$ and $\tilde{V}(s) = \tilde{X}(s) - Q(s)\tilde{N}(s)$ in equation (15) above, yields:

$$N(s)Q(s)\tilde{M}(s) - M(s)Q(s)\tilde{N}(s) = C(s), \tag{16}$$

$$C(s) = X(s)\tilde{M}(s) - M(s)\tilde{X}(s). \tag{17}$$

Finally, for $Q(s) = [\underline{q}_1(s) \quad \underline{q}_2(s) \quad \dots \quad \underline{q}_m(s)]$ and $C(s) = [\underline{c}_1(s) \quad \underline{c}_2(s) \quad \dots \quad \underline{c}_m(s)]$, it follows that equation (16) is equivalent to

$$P(s)\underline{q}(s) = \underline{c}(s), \tag{18}$$

$$\begin{aligned} P(s) &= \tilde{M}^t(s) \otimes N(s) - \tilde{N}^t(s) \otimes M(s) \\ \underline{q}(s) &= \left[\underline{q}_1^t(s) \quad \underline{q}_2^t(s) \quad \dots \quad \underline{q}_m^t(s) \right]^t, \\ \underline{c}(s) &= \left[\underline{c}_1^t(s) \quad \underline{c}_2^t(s) \quad \dots \quad \underline{c}_m^t(s) \right]^t \end{aligned} \tag{19}$$

with $\otimes$ denoting the Kronecker product. Equations (18) and (19) provide a necessary and sufficient condition for the existence of a rational stabilising commutative controller, namely that, there exists a rational $K(s)$ which stabilises the closed-loop system and commutes with $G(s)$ if and only if there exists a stable vector $\underline{q}(s) \in \mathbb{R}^{m^2}(s)$ such that equation (18) is satisfied.

Remark 1: Although $M(s)$, $N(s)$, $\tilde{M}(s)$, $\tilde{N}(s)$, $X(s)$ and $\tilde{X}(s)$ are rational, it is always possible to form these matrices in such a way that they all have the same denominator polynomial (Nett *et al.* 1984). Thus, it is always possible to assume that $P(s) \in \mathbb{R}^{m^2 \times m^2}[s]$ and $\underline{c}(s) \in \mathbb{R}^{m^2}[s]$.

3.2 *Existence of rational stabilising commutative controllers*

An RSCC $K(s)$ always exists when $G(s)$ is stable since in this case, it can be seen that

$$Q_e(s) = -M^{-1}(s)Y(s) = -\tilde{Y}(s)\tilde{M}^{-1}(s) \in RH_\infty^{m\times m} \tag{20}$$

and satisfies the commutativity conditions, equations (16) and (17). For unstable $G(s)$, $Q_e(s) \notin RH_\infty^{m\times m}$ so in general, one must characterise the space generated by all solutions to equation (18). Then writing

$$\underline{q}(s) = \frac{1}{d_q(s)}\underline{n}_q(s), \tag{21}$$

where $\underline{n}_q(s) \in \mathbb{R}^{m^2}[s]$, $d_q(s)$ is a polynomial, and substituting the $\underline{q}(s)$ of equation (21), in equation (18), yields:

$$P(s)\frac{1}{d_q(s)}\underline{n}_q(s) = \underline{c}(s), \tag{22}$$

or equivalently,

$$T(s)\begin{bmatrix} \underline{n}_q(s) \\ d_q(s) \end{bmatrix} = \underline{0}, \tag{23}$$

$$T(s) = \begin{bmatrix} P(s) & -\underline{c}(s) \end{bmatrix}. \tag{24}$$

Thus the solutions of equation (18) are defined by the right null space of $T(s)$ and are given by linear combinations of the elements of a minimal polynomial basis for the right null space of $T(s)$. The nullity of $T(s)$ can be deduced from the following result.

Lemma 2: Let $A \in \mathbb{C}^{m\times m}$ be a diagonalisable matrix and assume that each distinct eigenvalue of A, λ_k, $k = 1, 2, \dots, l$, have multiplicity μ_k. Then, there are $\Sigma_{k=1}^{l}\mu_k^2$ linearly independent matrices over the field of complex numbers ($\mathbb{C}$), which commutes with respect to multiplication with A.

Proof: Let $A = W\Lambda_A W^{-1}$ be a spectral decomposition of A, where $\Lambda_A = \mathrm{diag}\{\Lambda_{A_k}, k = 1, 2, \dots, l\}$, $\Lambda_{A_k} = \lambda_k I_{\mu_k}$, and consider a matrix B which commutes under multiplication with A. Therefore

$$W\Lambda_A W^{-1}B = BW\Lambda_A W^{-1}, \tag{25}$$

or equivalently:

$$\Lambda_A(W^{-1}BW) = (W^{-1}BW)\Lambda_A. \tag{26}$$

Denoting $\bar{B} = (W^{-1}BW)$ then, from equation (26), it can be seen that B commutes with A if and only if $\bar{B}$ commutes with Λ_A, i.e. if and only if $\bar{B}$ is block diagonal, $\bar{B} = \mathrm{diag}\{\bar{B}_k, k = 1, \dots, l\}$, where $\bar{B}_k \in \mathbb{C}^{\mu_k\times\mu_k}$. In addition, since Λ_{A_k} is a diagonal matrix, it commutes with any matrix $\bar{B}_k$. Let E_{ij} denote a matrix whose elements are zero except for element (i, j), which is equal to 1. It is immediate to see that all matrix $\bar{B}_k$ can be expressed as a linear combination of E_{ij}, i.e.

$$\bar{B}_k = \sum_{i=1}^{\mu_k}\sum_{j=1}^{\mu_k}\bar{b}_{ij}^{(k)}E_{ij}, \tag{27}$$

where $\bar{b}_{ij}^{(k)}$ is element (i, j) of $\bar{B}_k$, thus defining μ_k^2 linearly independent matrices that commute with Λ_{A_k}. Since there are l distinct blocks Λ_{A_k}, it follows that there are $\Sigma_{k=1}^{l}\mu_k^2$ linearly independent matrices that commute with A. □

From lemma 2 one can obtain the nullity of $P(s)$ and $T(s)$ from the eigenfunctions of $G(s)$.

Theorem 1: Let $G(j\omega)$ be diagonalisable for all $\omega \in \mathbb{R}$ and assume that $G(s)$ has l distinct eigenfunctions $\lambda_{g_i}(s)$ with multiplicity μ_i. Then, the normal rank of $P(s)$ is equal to $m^2 - \bar{\nu}$, where $\bar{\nu} = \Sigma_{i=1}^{l} \mu_i^2$. Furthermore, the nullity of $T(s)$ is $\bar{\nu} + 1$.

Proof: For $P(s)$ to have normal rank $r < m^2$, there must be $\bar{\nu} = m^2 - r$ linearly independent polynomial vectors $\underline{\alpha}(s) \in \mathbb{R}^{m^2}[s]$ such that:

$$\underline{\alpha}^t(s)P(s) = \underline{0}^t, \tag{28}$$

where $\underline{\alpha}^t(s) = [\, \underline{\alpha}_1^t(s) \quad \underline{\alpha}_2^t(s) \quad \ldots \quad \underline{\alpha}_m^t(s)]$, $\underline{\alpha}_i(s) \in \mathbb{R}^m[s]$. Defining,

$$A(s) = [\, \underline{\alpha}_1(s) \quad \underline{\alpha}_2(s) \quad \ldots \quad \underline{\alpha}_m(s)]^t, \tag{29}$$

then equation (28) holds true if there exists a matrix $A(s)$, formed according to equation (29) that satisfies:

$$\tilde{M}(s)A(s)N(s) - \tilde{N}(s)A(s)M(s) = O. \tag{30}$$

Premultiplying and postmultiplying equation (30) by $\tilde{M}^{-1}(s)$ and $M^{-1}(s)$, respectively, yields:

$$A(s)N(s)M^{-1}(s) - \tilde{M}^{-1}(s)\tilde{N}(s)A(s) = O. \tag{31}$$

Using $G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$ and $G(s) = N_G(s)/d(s)$, where $d(s)$ is the least common denominator of all the elements of $G(s)$ and $N_G(s) \in \mathbb{R}^{m\times m}[s]$, equation (31) can be rewritten as:

$$A(s)\frac{1}{d(s)}N_G(s) = \frac{1}{d(s)}N_G(s)A(s). \tag{32}$$

Since, by assumption, $G(s)$ has l distinct eigenfunctions $\lambda_{g_i}(s)$ with multiplicity μ_i, then for an infinite number of frequencies ω_k, $N_G(j\omega_k)$ has l distinct eigenvalues where each one has multiplicity μ_i. Let ω_k be such that $j\omega_k$ is not a zero of $d(s)$. Then equation (32) is satisfied if and only if $A(j\omega_k)$ commutes with $N_G(j\omega_k)$. According to lemma 2, there exist $\Sigma_{i=1}^{l}\mu_i^2$ linearly independent matrices that commute with $N_G(j\omega_k)$, hence for an infinite number of values of ω_k, the nullity of $P(j\omega_k)$ is $\bar{\nu} = \Sigma_{i=1}^{l}\mu_i^2$, or equivalently, the rank of $P(j\omega_k)$ is equal to $m^2 - \bar{\nu}$, and thus, according to definition 1, the normal rank of $P(s)$ is $m^2 - \bar{\nu}$.

Finally, notice that the commutativity condition given by equations (16) and (17) is always verified when $Q(s) = Q_e(s)$ given by equation (20), which implies that the vector $\underline{c}(s)$ always belongs to the space generated by the columns of $P(s)$. Therefore, assuming that the polynomial matrix $P(s)$ has nullity equal to $\bar{\nu}$, then $T(s)$, given by equation (24), has nullity $\bar{\nu} + 1$. □

According to theorem 1, the characterisation of all solutions to equation (23) is given by the right null space of $T(s)$ which has dimension $\bar{\nu} + 1$, and is spanned by the vectors of a minimal polynomial basis, defined by the columns of a $(m^2 + 1) \times (\bar{\nu} + 1)$ polynomial matrix, say $H(s)$ (i.e. $T(s)H(s) = O$). Then, all solutions to equation (23) should be of the following form:

$$\begin{bmatrix} \underline{n}_q(s) \\ d_q(s) \end{bmatrix} = H(s)\underline{\psi}(s), \tag{33}$$

where $\underline{\psi}(s)$ is a polynomial vector. Partitioning $H(s)$ as

$$H(s) = \begin{bmatrix} H_t(s) \\ \underline{h}_b^t(s) \end{bmatrix}, \tag{34}$$

then $d_q(s)$ can be written as:

$$d_q(s) = \sum_{i=1}^{\bar{\nu}+1} h_{b_i}(s)\psi_i(s), \tag{35}$$

which is a Diophantine equation. Thus equation (18) has a stable solution if and only if there exist polynomials $\psi_i(s)$, $i = 1, 2, \ldots, \bar{\nu} + 1$, such that $d_q(s)$ is a Hurwitz polynomial. However, such a solution, if it exists, may not lead to a proper controller since no constraints have been imposed on the degrees of $h_{b_i}(s)$ and the corresponding column vector of $H_t(s)$. Thus, the problem of finding a rational stabilising commutative controller for a given plant $G(s)$ turns out to be that of finding a Hurwitz polynomial $d_q(s)$ such that $Q(s)$ is proper. The following definition will be needed.

Definition 6: The degree of a polynomial vector $\underline{h}(s)$, denoted by $\deg[\underline{h}(s)]$, is equal to the largest degree of the polynomials of $\underline{h}(s)$.

The proof for the existence of RSCC will be carried out in two steps: first it will be shown that it is always possible to find $\psi_i(s)$, $i = 1, 2, \ldots, \bar{\nu} + 1$, such that the degree of $d_q(s)$ is greater than or equal to the degree of $\underline{n}_q(s)$ and that a Hurwitz $d_q(s)$ can always be obtained; then necessary and sufficient conditions on $G(s)$ for the existence of $\psi_i(s)$, such that $Q(s) \in RH_\infty$, will be given.

Consider first the problem of guaranteeing the existence of a proper $Q(s)$.

Lemma 3: Let $B(s)$ denote a polynomial matrix with linearly independent columns, and, for a polynomial vector $\underline{\psi}(s)$, define

$$\underline{p}(s) = B(s)\underline{\psi}(s). \tag{36}$$

Then, $B(s)$ is column reduced if and only if

$$\deg[\underline{p}(s)] = \max_{i:\psi_i(s)\neq 0} \{\deg[\underline{b}_i(s)] + \deg[\psi_i(s)]\} \tag{37}$$

where $\psi_i(s)$ and $\underline{b}_i(s)$ denote, respectively, the ith element of $\underline{\psi}(s)$ and the ith column of $B(s)$.

Proof: See Kailath (1980, p. 387). □

Notice that, since $H(s)$ is a minimal polynomial basis for the right null space of $T(s)$, it is a column reduced matrix. This leads to the following result.

Lemma 4: Let $H(s)$ be a minimal polynomial basis for the right null space of $T(s)$, defined according to equations (23) and (24). Then, at least one of the columns of $H(s)$, $h_k(s)$, is such that $\deg[\underline{h}_{t_k}(s)] \leq \deg[h_{b_k}(s)]$, for some $k \in \{1, 2, \ldots, \bar{\nu}+1\}$.

Proof: From equation (33), it can be seen that $\underline{n}_q(s)$ and $d_q(s)$ are obtained from a linear combination of the columns of $H(s)$. Thus, according to lemma 3,

$$\deg\left\{\begin{bmatrix} \underline{n}_q(s) \\ d_q(s) \end{bmatrix}\right\} = \max_{i:\psi_i(s)\neq 0}\{\deg[\underline{h}_i(s)] + \deg[\psi_i(s)]\}. \quad (38)$$

Assume that each element of $\underline{h}_b(s)$ has degree smaller than the degree of the corresponding column of $H_t(s)$, i.e. $\deg[h_{b_i}(s)] < \deg[\underline{h}_{t_i}(s)]$, for $i = 1, 2, \ldots, \bar{\nu}+1$. Then, $\deg[\underline{h}_i(s)] = \deg[\underline{h}_{t_i}(s)]$, which implies that equation (38) can be re-written as:

$$\deg\left\{\begin{bmatrix} \underline{n}_q(s) \\ d_q(s) \end{bmatrix}\right\} = \max_{i:\psi_i(s)\neq 0}\{\deg[\underline{h}_{t_i}(s)] + \deg[\psi_i(s)]\}. \quad (39)$$

On the other hand, it is immediate to see that

$$\deg[d_q(s)] \leq \max_{j:\psi_j(s)\neq 0}\{\deg[h_{b_j}(s)] + \deg[\psi_j(s)]\}. \quad (40)$$

According to equation (39), in order for $Q(s)$ to be proper, the degree of $d_q(s)$ must be given as:

$$\deg[d_q(s)] = \max_{i:\psi_i(s)\neq 0}\{\deg[\underline{h}_{t_i}(s)] + \deg[\psi_i(s)]\}. \quad (41)$$

Equations (40) and (41) lead to the following inequality:

$$\begin{aligned} &\max_{j:\psi_j(s)\neq 0}\{\deg[h_{b_j}(s)] + \deg[\psi_j(s)]\} \\ &\quad \geq \max_{i:\psi_i(s)\neq 0}\{\deg[\underline{h}_{t_i}(s)] + \deg[\psi_i(s)]\}. \end{aligned} \quad (42)$$

Assume that the maximum of the left-hand side of (42) has been attained for $i = k$ so that:

$$\begin{aligned} \deg[h_{b_k}(s)] + \deg[\psi_k(s)]\} &\geq \max_{i:\psi_i(s)\neq 0}\{\deg[\underline{h}_{t_i}(s)] \\ &+ \deg[\psi_i(s)]\} \geq \deg[\underline{h}_{t_k}(s)] + \deg[\psi_k(s)], \end{aligned} \quad (43)$$

or, equivalently, $\deg[h_{b_k}(s)] \geq \deg[\underline{h}_{t_k}(s)]$. Note that, this inequality never holds true since, by assumption, $\deg[h_{b_i}(s)] < \deg[\underline{h}_{t_i}(s)]$, for $i = 1, 2, \ldots, \bar{\nu}+1$. Thus, when $\deg[h_{b_i}(s)] < \deg[\underline{h}_{t_i}(s)]$, for $i = 1, 2, \ldots, \bar{\nu}+1$, then all vectors $\begin{bmatrix} \underline{n}_q^t(s) & d_q(s) \end{bmatrix}^t$ that satisfy equation (33), must be such that $\deg[d_q(s)] < \deg[\underline{n}_q(s)]$, leading to non-proper matrices $Q(s)$. However, for any given $G(s)$, $Q_e(s)$ is proper and satisfies the commutativity condition (33) which contradicts the fact that $\deg[h_{b_i}(s)] < \deg[\underline{h}_{t_i}(s)]$, for $i = 1, 2, \ldots, \bar{\nu}+1$. Therefore, at least one column of $H(s)$ must satisfy the condition $\deg[h_{b_k}(s)] \geq \deg[\underline{h}_{t_k}(s)]$. □

The consequence of lemma 4, is that, it is always possible to obtain a Hurwitz polynomial $d_q(s)$ such that $\deg[\underline{n}_q(s)] \leq \deg[d_q(s)]$, as follows.

Theorem 2: Let $h_{b_1}(s), h_{b_2}(s), \ldots, h_{b_{\bar{\nu}+1}}(s)$ be the polynomials of $\underline{h}_b^t(s)$, and assume, without loss of generality, that they are coprime. In addition, assume that $\deg[h_{b_i}(s)] \leq \deg[\underline{h}_{t_i}(s)]$ and that for some $k \in \{1, 2, \ldots, \bar{\nu}+1\}$, $\deg[\underline{h}_{t_k}(s)] = \deg[h_{b_k}(s)]$. Let $\psi_1(s), \psi_2(s), \ldots, \psi_{\bar{\nu}+1}(s)$ be the polynomials to be determined and let $d_q(s)$ be defined by the following Diophantine equation:

$$d_q(s) = h_{b_1}(s)\psi_1(s) + h_{b_2}(s)\psi_2(s) + \cdots + h_{b_{\bar{\nu}+1}}(s)\psi_{\bar{\nu}+1}(s). \quad (44)$$

Then, if the degree of $\psi_k(s)$ is chosen such that $\deg[d_q(s)] = \deg[h_{b_k}(s)\psi_k(s)]$, it is always possible to find $\psi_i(s)$, $i = 1, 2, \ldots, \bar{\nu}+1$, such that:

(i) $d_q(s)$ is a Hurwitz polynomial;

(ii) $\deg\left\{\begin{bmatrix} \underline{n}_q(s) \\ d_q(s) \end{bmatrix}\right\} = \deg[d_q(s)]$.

Proof: Consider the Diophantine equation formed with $h_{b_k}(s)$ and $h_{b_i}(s)$, $i \neq k$, given by:

$$d_{ki}(s) = h_{b_k}(s)\bar{\psi}_k(s) + h_{b_i}(s)\bar{\psi}_i(s), \quad (45)$$

where $\bar{\psi}_k(s)$ and $\bar{\psi}_i(s)$ are polynomials such that $\deg[\bar{\psi}_k(s)] \geq \deg[\bar{\psi}_i(s)]$ and define:

$$\left\{\begin{aligned} h_{b_k}(s) &= h_{b_k}^{(0)}s^{\beta_k} + h_{b_k}^{(1)}s^{\beta_k-1} + \cdots + h_{b_k}^{(\beta_k)} \\ h_{b_i}(s) &= h_{b_i}^{(0)}s^{\beta_i} + h_{b_i}^{(1)}s^{\beta_i-1} + \cdots + h_{b_i}^{(\beta_i)} \\ \bar{\psi}_k(s) &= \bar{\psi}_k^{(0)}s^{\alpha_k} + \bar{\psi}_k^{(1)}s^{\alpha_k-1} + \cdots + \bar{\psi}_k^{(\alpha_k)} \\ \bar{\psi}_i(s) &= \bar{\psi}_i^{(0)}s^{\alpha_i} + \bar{\psi}_i^{(1)}s^{\alpha_i-1} + \cdots + \bar{\psi}_i^{(\alpha_i)} \end{aligned}\right. . \quad (46)$$

It will be shown that it is always possible to choose α_k and α_i (the degrees of $\bar{\psi}_k(s)$ and $\bar{\psi}_i(s)$) such that $\deg[d_{ki}(s)] = \deg[h_{b_k}(s)\bar{\psi}_k(s)] \geq \deg[h_{b_i}(s)\bar{\psi}_i(s)]$ and $d_{ki}(s)$ a Hurwitz polynomial. To do so, note that, for $\alpha_k + \beta_k \geq \alpha_i + \beta_i$, then solving equation (45) is equivalent to solving:

$$A\underline{\bar{\psi}}_{ki} = \underline{d}_{ki}, \quad (47)$$

where

$$A=\begin{bmatrix} h_{b_k}^{(0)} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ h_{b_k}^{(1)} & h_{b_k}^{(0)} & \ddots & 0 & 0 & 0 & \ddots & 0 \\ \vdots & h_{b_k}^{(1)} & \ddots & \vdots & h_{b_i}^{(0)} & 0 & \ddots & 0 \\ h_{b_k}^{(\beta_k)} & \vdots & \ddots & h_{b_k}^{(0)} & h_{b_i}^{(1)} & h_{b_i}^{(0)} & \ddots & \vdots \\ 0 & h_{b_k}^{(\beta_k)} & \ddots & h_{b_k}^{(1)} & \vdots & h_{b_i}^{(1)} & \ddots & 0 \\ 0 & 0 & \ddots & \vdots & h_{b_i}^{(\beta_i)} & \vdots & \ddots & h_{b_i}^{(0)} \\ 0 & 0 & \ddots & h_{b_k}^{(\beta_k-2)} & 0 & h_{b_i}^{(\beta_i)} & \ddots & h_{b_i}^{(1)} \\ \vdots & \vdots & \ddots & h_{b_k}^{(\beta_k-1)} & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & h_{b_k}^{(\beta_k)} & 0 & 0 & \cdots & h_{b_i}^{(\beta_i)} \end{bmatrix},$$

$$\underline{\bar{\psi}}_{ki}=\begin{bmatrix} \bar{\psi}_k^{(0)} \\ \bar{\psi}_k^{(1)} \\ \vdots \\ \bar{\psi}_k^{(\alpha_k)} \\ \bar{\psi}_i^{(0)} \\ \bar{\psi}_i^{(1)} \\ \vdots \\ \bar{\psi}_i^{(\alpha_i)} \end{bmatrix} \tag{48}$$

and $\underline{d}_{ki}$ is the vector formed by the coefficients of the polynomial $d_{ki}(s)$. Since A has dimension $(\alpha_k+\beta_k+1)\times(\alpha_k+\alpha_i+2)$, it will be a square matrix if and only if $\alpha_i=\beta_k-1$. Moreover, if the degree of $\bar{\psi}_k(s)$, α_k, is chosen to be greater than or equal to β_i-1, it can be checked that A will be a square matrix with the following structure:

$$A=\begin{bmatrix} L & O \\ B & S \end{bmatrix}, \tag{49}$$

where L is square triangular with diagonal elements equal to $h_{b_k}^{(0)}$, O the $(\alpha_k+\beta_k-\alpha_i-\beta_i)\times\beta_k$ zero matrix, and S is the Sylvester matrix formed with the coefficients of the polynomials $h_{b_k}(s)$ and $h_{b_i}(s)$. Since, by assumption, $h_{b_k}(s)$ and $h_{b_i}(s)$ are coprime, then A is non-singular, which implies that any Hurwitz polynomial with degree equal to the sum of the degrees of $h_{b_k}(s)$ and $\bar{\psi}_k(s)$ can be chosen arbitrarily. However, the properness of $Q(s)$ has not been guaranteed yet. To do so, note that the degree of $\bar{\psi}_i(s)(\alpha_i)$ is fixed $(\alpha_i=\beta_k-1)$, whereas the degree of $\bar{\psi}_k(s)$ (α_k) can be any integer greater than or equal to β_i-1. Let $\gamma_l=\deg[h_{t_l}(s)]$, $l=1,2,\ldots,\bar{\nu}+1$. Since, by assumption, $\gamma_i\geq\beta_i$, choosing $\alpha_k=\gamma_i-1$, it follows that:

$$\begin{cases} \deg[\underline{h}_{t_i}(s)\bar{\psi}_i(s)+\underline{h}_{t_k}(s)\bar{\psi}_k(s)]\leq\max\{\gamma_k+\alpha_k,\gamma_i+\alpha_i\} \\ \quad =\max\{\gamma_k+\gamma_i-1,\gamma_i+\beta_k-1\}=\gamma_k+\gamma_i-1 \\ \deg[d_{ki}(s)]=\beta_k+\alpha_k=\gamma_k+\gamma_i-1, \end{cases} \tag{50}$$

thus, leading to a $Q(s)\in RH_\infty^{m\times m}$, formed by equation (33) from a linear combination of $\underline{h}_i(s)$ and $\underline{h}_k(s)$.

To show that the $d_q(s)$ of (44) can always be Hurwitz, note that for all $j\neq k$, $j\neq i$, one can form a new Diophantine equation $d_{kij}(s)=d_{ki}(s)\bar{\psi}_{ki}(s)+h_{b_j}(s)\bar{\psi}_j(s)$. The procedure above can be repeated with $d_{ki}(s)$ and $h_{b_j}(s)$ to guarantee that the degree of $d_{kij}(s)$ is equal to the sum of the degrees of $d_{ki}(s)$ and $\bar{\psi}_{ki}(s)$, leading to a Hurwitz polynomial $d_q(s)$ and to a proper $Q(s)$. Then

$$d_{kij}(s)=h_{b_k}(s)\bar{\psi}_k(s)\bar{\psi}_{ki}(s)+h_{b_i}(s)\bar{\psi}_i(s)\bar{\psi}_{ki}(s)+h_{b_j}(s)\bar{\psi}_j(s), \tag{51}$$

and, by construction, $\deg[d_{kij}(s)]=\deg[h_{b_k}(s)]+\deg[\bar{\psi}_k(s)]+\deg[\bar{\psi}_{ki}(s)]$. Repeating the procedure above for all the elements of $\underline{h}_b(s)$, leads to the desired result. □

Remark 2: Theorem 2 above presents a systematic manner to obtain a Hurwitz polynomial $d_q(s)$ such that

$$\deg\left\{\begin{bmatrix} \underline{n}_q(s) \\ d_q(s) \end{bmatrix}\right\}=\deg[d_q(s)],$$

leading, therefore to a matrix $Q(s)\in RH_\infty^{m\times m}$. In order to illustrate this point, assume that $\bar{\nu}=1$ and that the column degrees of $H_t(s)$ and $\underline{h}_b^t(s)$ are given as:

$$\begin{bmatrix} \deg[\underline{h}_{t_1}(s)] & \deg[\underline{h}_{t_2}(s)] \\ \deg[h_{b_1}(s)] & \deg[h_{b_2}(s)] \end{bmatrix}=\begin{bmatrix} 4 & 7 \\ 4 & 2 \end{bmatrix}, \tag{52}$$

where $h_{b_1}(s)$, $h_{b_2}(s)$ are coprime. Proceeding as in theorem 2, one has to choose $\deg[\psi_2(s)]=\deg[h_{b_1}(s)]-1=3$, $\deg[\psi_1(s)]\geq\deg[h_{b_2}(s)]-1=1$. Thus, since $\deg[\underline{h}_{t_2}(s)]>\deg[h_{b_2}(s)]$, choosing $\deg[\psi_1(s)]=\deg[\underline{h}_{t_2}(s)]-1=6$, it is clear that $\deg[\underline{n}_q(s)]\leq 10$, and $\deg[d_q(s)]=10$.

A consequence of theorem 2 is that the problem of finding a rational stabilising commutative controller for a given plant $G(s)$ turns out to be that of finding a Hurwitz polynomial $d_q(s)$. Notice that, according to equation (35), such a polynomial always exists if and only if $h_{b_i}(s)$, $i=1,2,\ldots,\bar{\nu}+1$ do not have a common unstable zero. An important result that relates the poles of the plant TFM to the vector $\underline{h}_b^t(s)$, defined in equation (34), is now presented. This result will be used in the sequel to obtain a necessary and sufficient condition for the existence of RSCCs.

Lemma 5: If $\underline{h}_b^t(s_0) = \underline{0}^t$, for some $s_0 \in \mathbb{C}$, then s_0 must be a pole of $G(s)$.

Proof: If $\underline{h}_b^t(s_0) = \underline{0}^t$ for some $s_0 \in \mathbb{C}$, then s_0 will be a pole of any particular solution to equations (16) and (17). Consider now the particular solution $Q_e(s) = -M^{-1}(s)Y(s)$. It can be easily verified that $Q_e(s)$ has the same poles as $G(s)$, and thus, forming the vector $\underline{q}_e(s) = (1/d_{q_e}(s))\,\underline{n}_{q_e}(s)$, in accordance with equation (21), it can be concluded that the vector $[\underline{n}_{q_e}^t(s) \quad d_{q_e}(s)]^t$ satisfies equation (23) and thus $d_{q_e}(s_0) = 0$, which means that s_0 must be a pole of $G(s)$. □

A necessary and sufficient condition for the existence of RSCC follows from lemma 5.

Theorem 3: Let $G(s)$ be the plant TFM. Then, there exist RSCCs for $G(s)$ if and only if $\underline{h}_b^t(s_0) \neq \underline{0}^t$, for all s_0 equal to an unstable pole of the plant.

Proof: Note that there does not exist $\psi_i(s)$, $i = 1, 2, \ldots, \bar{\nu} + 1$, such that $d_q(s)$ is a Hurwitz polynomial if and only if the greatest common divisor of $h_{b_i}(s)$, for $i = 1, 2, \ldots, \bar{\nu} + 1$, $\chi(s)$, is such that $\chi(s_0) = 0$ for $s_0 \in \mathbb{C}^+$, where $\mathbb{C}^+ = \{s \in \mathbb{C} : \mathrm{Re}(s) \geq 0\}$. Thus there does not exist any RSCC if and only if $\underline{h}_b^t(s_0) = \underline{0}^t$ and $s_0 \in \mathbb{C}^+$. According to lemma 5, if $\underline{h}_b^t(s_0) = \underline{0}^t$ then s_0 must be a plant pole, and so an RSCC will not exist if and only if $\underline{h}_b^t(s_0) = \underline{0}^t$ for an unstable pole s_0 of the plant. □

Remark 3: Notice that when $\underline{h}_b^t(s_0) = \underline{0}^t$, then all $Q(s)$, that satisfies the commutativity condition given by equations (16) and (17), must have the unstable pole of $G(s)$, s_0, as a pole.

The necessary and sufficient condition for the existence of RSCCs of theorem 3 is based on $\underline{h}_b(s)$ which does not bear a direct relation to properties of the plant. Furthermore, implementation is demanding: it is necessary to compute a minimal polynomial basis for the right null space of $T(s)$ ($H(s)$), and then to check if $\underline{h}_b^t(s_0) = \underline{0}^t$ for each unstable pole s_0 of the plant. Therefore, it would be more useful to find a condition for the existence of RSCC, based solely on the plant TFM and to achieve this, it is first necessary to present the following result.

Lemma 6: Let $s_0 \in \mathbb{C}^+$. Then s_0 is a zero of the closed-loop characteristic polynomial if and only if s_0 is a pole of $Q(s)$, the free parameter of the Youla–Kucera parameterisation given by equation (13), associated with a non-stabilising controller $K(s)$.

Proof: Let $K(s)$ be the TFM of a non-stabilising controller, and $Q(s)$ the TFM associated (by equation 13) with this $K(s)$. It is know that the return difference matrix and its determinant are given as:

$$F(s) = I_m + G(s)K(s), \quad \det[F(s)] = \alpha\frac{p_c(s)}{p_o(s)}, \tag{53}$$

where $\alpha \in \mathbb{R}$ and $p_c(s)$ and $p_o(s)$ denote, respectively, the closed-loop and open-loop characteristic polynomials. Let $G(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$ be a left coprime factorisation of $G(s)$ in $RH_\infty^{m\times m}$ and let $K(s) = \hat{U}(s)\hat{V}^{-1}(s)$ be obtained according to equation (13). Thus,

$$\begin{aligned}
\det[F(s)] &= \det\left[I_m + \tilde{M}^{-1}(s)\tilde{N}(s)\hat{U}(s)\hat{V}^{-1}(s)\right] \\
&= \det\left[\tilde{M}^{-1}(s)\right]\det\left[\tilde{M}(s)\hat{V}(s) + \tilde{N}(s)\hat{U}(s)\right] \\
&\quad \times \det\left[\hat{V}^{-1}(s)\right] \\
&= \det\left[\tilde{M}^{-1}(s)\right]\det\left[\hat{V}^{-1}(s)\right].
\end{aligned} \tag{54}$$

Defining $p_G(s)$ and $p_K(s)$ as the pole polynomials of $G(s)$ and $K(s)$, respectively, and $p_{\tilde{M}}(s)$ and $p_{\hat{V}}(s)$ as the pole polynomials of $\tilde{M}(s)$ and $\hat{V}(s)$, and combining equations (54) and (53), yields:

$$\frac{p_G(s)p_K(s)}{p_{\tilde{M}}(s)p_{\hat{V}}(s)} = \frac{p_o(s)}{p_c(s)}. \tag{55}$$

Assuming that $G(s)$ and $K(s)$ have no decoupled modes, then equation (55) leads to

$$p_c(s) = p_{\tilde{M}}(s)p_{\hat{V}}(s). \tag{56}$$

Notice that, since $\tilde{M}(s) \in RH_\infty^{m\times m}$, then $p_{\tilde{M}}(s)$ is a Hurwitz polynomial and thus, since by assumption the closed-loop system is unstable, then the unstable zeros of $p_c(s)$ must be zeros of $p_{\hat{V}}(s)$, or equivalently, unstable poles of $\hat{V}(s)$. According to equation (13),

$$\hat{V}(s) = X(s) - N(s)Q(s), \tag{57}$$

But since $X(s), N(s) \in RH_\infty^{m\times m}$, all unstable poles of $\hat{V}(s)$ must also be poles of $Q(s)$. Conversely, if $Q(s)$ has an unstable pole then the closed-loop system is unstable, i.e. $p_c(s)$ is not Hurwitz. □

Lemma 6 associates the non-existence of stabilising controllers to the presence of unstable poles of $Q(s)$ and theorem 3 (see remark 3) relates the non-existence of RSCC to unstable poles of $G(s)$ which are also poles of $Q(s)$. The exact characterisation of such poles is given below.

Theorem 4: No RSCC exist if and only if $G(s)$, the plant TFM, has at least one unstable fixed mode.

Proof: ($\Leftarrow$) Suppose that $G(s)$ has an unstable fixed mode s_0 and assume that $K(s)$ commutes exactly with $G(s)$. Therefore, $K(s)$ and $G(s)$ share the same eigenvector matrix $W(s)$, i.e. if $G(s) = W(s)\Lambda_G(s)V(s)$, $V(s) = W^{-1}(s)$, is a spectral decomposition of $G(s)$, then $K(s) = W(s)\Lambda_K(s)V(s)$ is a spectral decomposition of $K(s)$. Consider the open-loop transfer matrix $T_o(s) = G(s)K(s)$. Thus, a spectral decomposition of $T_o(s)$ is given by

$$T_o(s) = W(s)\Lambda_G(s)\Lambda_K(s)V(s), \tag{58}$$

which shows that the eigenfunctions of $T_o(s)$ are the product of the eigenfunctions of $G(s)$ and $K(s)$. Therefore the poles of the eigenfunctions of $T_o(s)$ are the poles of the eigenfunctions of $G(s)$ and $K(s)$. However, since s_0 is a fixed mode of $G(s)$, s_0 is a pole of $G(s)$ that is not a pole of any of its eigenfunctions, which implies that s_0 cannot be a pole of the eigenfunctions of $T_o(s)$. Hence, there are two possibilities: (*i*) s_0 is a pole of $T_o(s)$, i.e. s_0 is a fixed mode of $T_o(s)$ and, by lemma 1, it is a zero of $p_c(s)$; (*ii*) s_0 is not a pole of $T_o(s)$, which means that, since s_0 is a pole of $G(s)$, it must be a decoupled mode of $T_o(s)$, which ultimately implies that s_0 is a zero of $p_c(s)$.

($\Rightarrow$) Assume now that $G(s)$ has no unstable fixed modes and that there does not exist any RSCC for $G(s)$. According to theorem 3, there does not exist any RSCC for $G(s)$ if and only if an unstable pole of $G(s)$ is also a pole of all $Q(s)$ that satisfies the commutativity condition given by equations (16) and (17). Furthermore, using lemma 6, it can be seen that there is no RSCC for $G(s)$ if an unstable pole of $Q(s)$, and consequently of $G(s)$, is also a zero of $p_c(s)$. Let s_0 be an unstable pole of the plant that is also a zero of $p_c(s)$. Therefore, no matter what commutative controller $K(s)$ is obtained, the closed-loop system will be unstable and will have s_0 as a pole. Consider, then, the following commutative controller: $K(s) = kI_m$, $k \in \mathbb{R}$. This controller also makes s_0 unaltered and thus, according to lemma 1, s_0 is a fixed mode of $G(s)$, which contradicts the assumption that $G(s)$ has no unstable fixed modes. □

3.3 *General solution and characterisation of the degrees of freedom*

The general solution to the problem of finding a polynomial matrix $Q(s) \in RH_\infty^{m\times m}$, which leads to a rational stabilising commutative controller $K(s)$, is now presented.

Theorem 5: Suppose that $G(s) \in \mathbb{R}^{m\times m}(s)$ satisfies the conditions given by theorem 4 for the existence of an RSCC. Then, the class of all RSCC can be parameterised by a rational, proper and stable transfer matrix $Q(s)$ whose columns $q_i(s)$, $i = 1, 2, \ldots, m$, are obtained as follows:

$$\underline{q}(s) = \begin{bmatrix} \underline{q}_1(s) \\ \underline{q}_2(s) \\ \vdots \\ \underline{q}_m(s) \end{bmatrix} = \frac{1}{d_q(s)} H_t(s)\underline{\psi}(s) \tag{59}$$

where

(i) $H(s) = \left[\begin{smallmatrix} H_t(s) \\ \underline{h}_b^t(s) \end{smallmatrix}\right]$ is a $(m^2+1)\times(\bar{\nu}+1)$ polynomial matrix whose columns form a minimal polynomial basis for the right null space of the matrix $T(s) = \begin{bmatrix} P(s) & -\underline{c}(s) \end{bmatrix}$ defined in equation (24);

(ii) $\underline{\psi}(s)$ is a $(\bar{\nu}+1)$-dimensional vector whose entries are polynomials, being the degrees of freedom available on the general solution, which are deployed to obtain a Hurwitz polynomial $d_q(s) = \Sigma_{i=1}^{\bar{\nu}+1} h_{b_i}(s)\psi_i(s)$, where $h_{b_i}(s)$, for $i = 1, \ldots, \bar{\nu}+1$, are the entries of vector $\underline{h}_b(s)$.

4. Numerical example

The degrees of freedom in the parameterisation of the RSCC, given in this paper, can be used to advantage as will now be illustrated by means of an example. Thus consider the TFM

$$G(s) = \frac{1}{d_G(s)} \begin{bmatrix} -47s+2 & 56s \\ -42s & 50s+2 \end{bmatrix}, \tag{60}$$

where $d_G(s) = (s-1)(s+2)$, for which CLM is known to have sensitivity problems (see Doyle and Stein (1981)). To make the problem more challenging, here an unstable pole has been added. The design specifications are: (S1) rise times of no more than 2.5 s for each loop; (S2) low interactions between outputs; (S3) good damping of step responses with peak overshoot of no more than about 40%; (S4) zero steady-state error for step reference input and disturbance; and (S5) the feedback system is required to tolerate multiplicative uncertainty up to $1/\sqrt{2}$ which means that

$$\|T_C\|_\infty < \frac{1}{\delta_G} = \sqrt{2}, \tag{61}$$

where $T_C(s)$ denote the closed-loop TFM of the system shown in figure 1.

For all frequencies, except at $\omega = 0$ rad/s, the condition number of the eigenvector matrix of $G(s)$ is approximately 196, thereby indicating significant CLM sensitivity problems. To avoid these, a normalising precompensator is designed (Moreira and Basilio (2005)):

$$K_P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{62}$$

and the precompensated plant TFM becomes:

$$G_P(s) = G(s)K_P = \frac{1}{(s-1)(s+2)}\begin{bmatrix} -56s & -47s+2 \\ -(50s+2) & -42s \end{bmatrix}. \tag{63}$$

The efficacy of this K_P is illustrated by the condition number of the eigenvector matrix of $G_P(s)$ which is approximately equal to 1 for all frequencies.

According to theorem 4, an RSCC for $G_P(s)$ exists if and only if $G_P(s)$ has no unstable fixed modes. The Smith–McMillan form of $G_P(s)$ shows that $G_P(s)$ has two unstable poles equal to 1. Since this pole is not a zero of $G_P(s)$, then this unstable pole cannot be a fixed mode. Therefore, the condition of theorem 4 is satisfied and the existence of RSCC for $G_P(s)$ is guaranteed. To obtain a parameterisation of all RSCC for $G_P(s)$, it is necessary to compute a doubly coprime factorisation in $RH_\infty^{m\times m}$ for $G_P(s)$ and to form, according to equation (19), the polynomial matrix $P(s)$ and the polynomial vector $\underline{c}(s)$. From these one can form the matrix $T(s) = \begin{bmatrix} P(s) & -\underline{c}(s) \end{bmatrix}$ and compute a minimal polynomial basis $H(s)$ for the right null space of $T(s)$. Since $G_P(s)$ has two distinct eigenfunctions, the nullity of $P(s)$ is $\bar{\nu} = 2$, and by theorem 1, the nullity of $T(s)$ is $\bar{\nu} + 1 = 3$, so the computation of $H(s)$ requires the determination of 3 polynomial vectors. Using the algorithm proposed in Basilio and Moreira (2004) to compute $H(s)$, one obtains:

$$H(s) = \begin{bmatrix} -0.4824 & 0.1552s + 0.4077 & 0.3992s - 0.1471 \\ 0.5100 & -0.1849s - 0.4824 & -0.4753s + 0.1758 \\ 0.4794 & -0.1738s - 0.4423 & -0.4467s + 0.1805 \\ -0.5712 & 0.2070s + 0.5234 & 0.5322s - 0.2160 \\ 0.0148 & 0.0007s - 0.0011 & -0.0107s + 0.0090 \end{bmatrix}. \tag{64}$$

According to theorem 5, the class of all RSCCs for $G_P(s)$ can be parameterised by the matrix $Q(s)$, obtained from $H(s)$ and with the degrees of freedom, $\underline{\psi}(s)$, chosen such that

$$d_q(s) = 0.0148\psi_1(s) + (0.0007s - 0.0011)\psi_2(s) + (-0.0107s + 0.0090)\psi_3(s) \tag{65}$$

is a Hurwitz polynomial.

The next step is to deploy the degrees of freedom given by $\psi_1(s)$, $\psi_2(s)$ and $\psi_3(s)$ to derive a particular RSCC, $K_C(s)$, that meets the specifications S1–S5. Note that S4 requires the use of integral action and hence $K_C(s)$ will be replaced by $K_C(s)K_I(s)$, where $K_I = [k(s+1/\alpha)/s]I$ and where (as is done in the classical frequency response approach) α can be used to reach a trade off between bandwidth and relative stability margins. $K_C(s)K_I(s)$ clearly commutes with $G_P(s)$ and thus, provided that $T_o(s) = G_P(s)K_C(s)K_I(s)$ satisfies the generalised Nyquist stability criterion, $K_C(s)K_I(s)$ itself will be an RSCC. Note next that $G_P(s)$ is near normal and therefore (due to commutativity) so will $T_o(s)$ be, so that S5 will be met if and only if the characteristic loci of $T_o(s)$ do not intersect M-circles with $M \geq \sqrt{2}$. This condition, according to the rules of thumb used in classical frequency response (which are generalised in a natural way to the multivariable case through the use of the characteristic loci), will also ensure satisfaction of S3. Selecting $d_q(s) = s + \beta$ and using a simple trial-and-error search over β shows that S4 and S5 are satisfied for $\beta = 15$ and gives $\psi_1 = 1064.8$, $\psi_2 = -109.5$ and $\psi_3 = -100$, and $Q(s) \in RH_\infty^{m\times m}$:

$$Q(s) = \frac{1}{s+15}\begin{bmatrix} -56.92s - 486.1 & 63.71s + 540.8 \\ 67.78s + 578.3 & -75.9s - 644 \end{bmatrix}. \tag{66}$$

Substituting $Q(s)$, given by equation (66), in the Youla–Kucera parameterisation (equation (13)), yields:

$$K_C(s) = N_{K_C}(s)M_{K_C}^{-1}(s) \tag{67}$$

where

$$N_{K_C}(s) = \begin{bmatrix} -19.72s^2 - 184.24s - 505.12 & 6.68s^3 + 77.85s^2 + 307.84s + 368.44 \\ 23.15s^2 + 209.45s + 577.78 & -7.75s^3 - 85.82s^2 - 341.60s - 414.28 \end{bmatrix} \tag{68}$$

$$M_{K_C}(s) = \begin{bmatrix} 16.88s^2 + 287.35s + 324.54 & -10.71s^3 - 194.42s^2 - 411.79s - 232.70 \\ 14.77s^2 + 251.96s + 283.54 & -9.46s^3 - 172.04s^2 - 365.42s - 206.81 \end{bmatrix}. \tag{69}$$

The high degree of commutativity between $G_P(j\omega)$ and $K_C(j\omega)$ (and hence $K_C(j\omega)K_I(j\omega)$) is established by the very low values of the indicator:

$$e_i(\omega)(\%) = \frac{\left|\lambda_{o_i}(j\omega) - \lambda_{G_{Pi}}(j\omega)\lambda_{K_{Ci}}(j\omega)\right|}{\left|\lambda_{o_i}(j\omega)\right|}100\%, \quad i = 1, 2, \tag{70}$$

which here, due to rounding, is non-zero but less than 0.1% over all frequencies.

The two characteristic loci for $G_P(s)K_C(s)K_I(s)$ for $k = 2$ and $\alpha = 0.2$ (both values chosen as in the classical frequency approach) are shown in figure 2 and can be seen to give the two anticlockwise encirclements (required for closed-loop stability) and to intersect the $M = 1$ circle (namely the straight line which is perpendicular to the real axis and passes through $-0.5 + j0$) at $\omega = 2.6$ and $\omega = 3$ rad/s thereby indicating closed-loop rise times of about 0.3 and 0.4 (both of which are within the specification S1). This is also confirmed by the simulated closed-loop step responses shown in figure 3 which also exhibit low interaction and well-damped responses whose peak overshoot is about 40%. The latter is a consequence of the fact that both characteristic loci do not intersect M-circles with M greater than about 1.46 (which by the classical frequency approach approximation which generalizes exactly to the multivariable case) would predict overshoots of no more than 47%. It is known that closed-loop low interaction is a consequence of any of three conditions (MacFarlane and Kouvaritakis 1977): characteristic loci of large moduli; low misalignment angles;

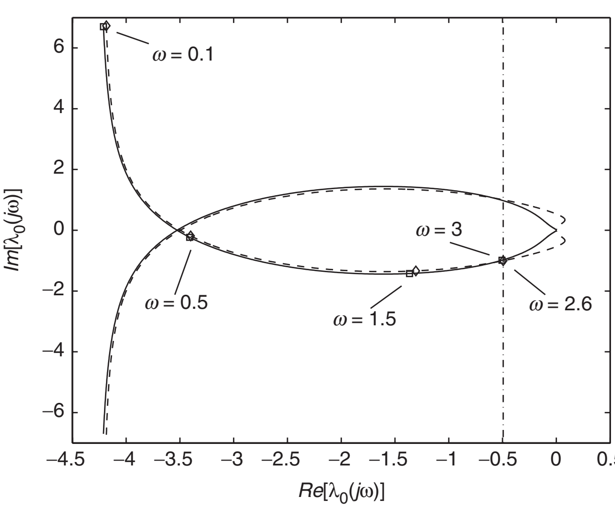

Figure 2. Characteristic loci of the open-loop transfer matrix $G(s)K(s)$, (solid and dashed lines), and the M-circle for $M = 1$ (dash-dotted line).

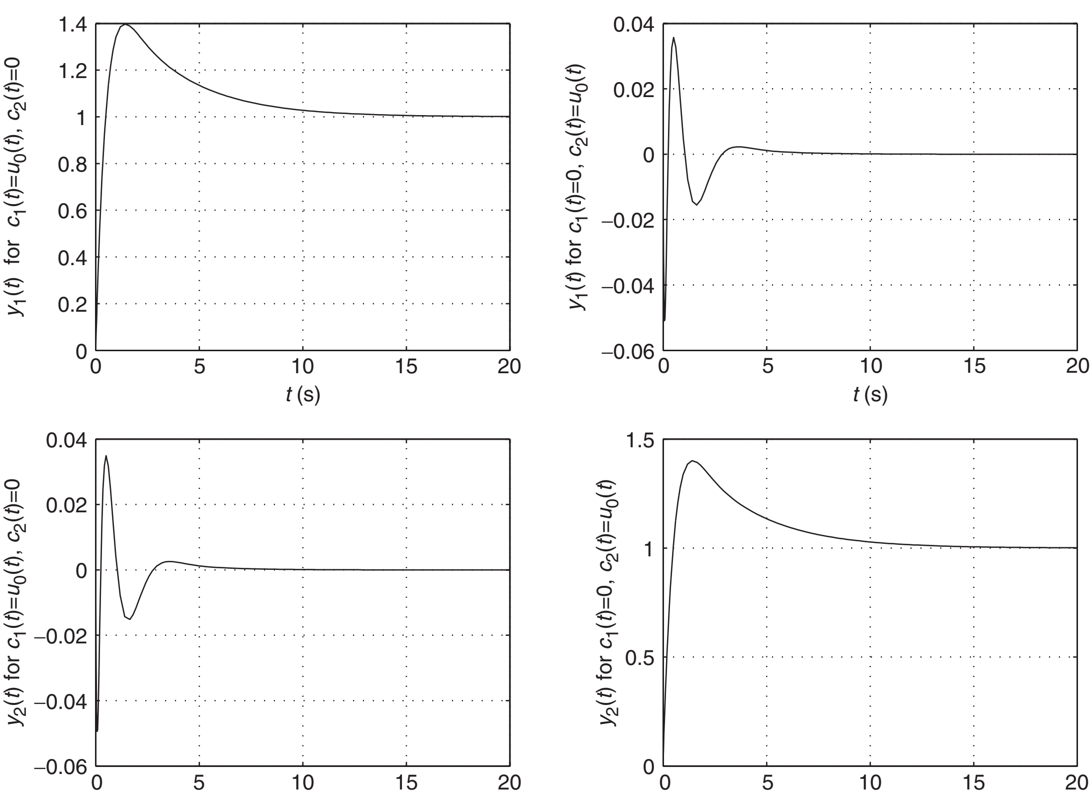

Figure 3. Step responses of the closed-loop system, where $u_0(t)$ denotes the unit step signal.

approximately equal characteristic loci. Here, the latter condition holds over most of the characteristic loci bandwidths whereas at low frequencies, integral action ensure the presence of large modulus characteristic loci.

5. Conclusions

In this paper a parameterisation of all rational stabilising commutative controllers for continuous time systems is presented. In addition, a complete characterisation of the degrees of freedom available in this parameterisation is given. A necessary and sufficient condition for the existence of rational stabilising commutative controllers for unstable plants is also presented.

The example used in the paper to illustrate the parameterisation of all stabilising commutative controllers suggests that there are sufficient degrees of freedom in this parameterisation in order to consider, besides stability, other control objectives concerning robustness and dynamic behaviour.

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References

J.C. Basilio and B. Kouvaritakis, "The use of rational eigenvector approximations in commutative controllers", *Inter. J. Cont.*, 61, pp. 333–356, 1995.

J.C. Basilio and M.V. Moreira, "A robust solution of the generalized polynomial Bezout identity", *Linear Algebra and Its Appli.*, 385, pp. 287–303, 2004.

J.C. Basilio and J.A. Sahate, "A normalizing precompensator for the design of effective and reliable commutative controllers", *Inter. J. Cont.*, 73, pp. 1280–1297, 2000.

R. Cameron and B. Kouvaritakis, "Relative stability margins of multivariable systems – characteristic locus approach", *Inter. J. Cont.*, 30, pp. 629–651, 1979.

J.C. Doyle and G. Stein, "Multivariable feedback design: concepts for a classic/modern synthesis", *IEEE-Trans. on Autom. Cont.*, 26, pp. 4–16, 1981.

D.G. Forney, "Minimal bases of rational vector spaces, with applications to multivariable linear systems", *SIAM J. Cont.*, 13, pp. 493–520, 1975.

T. Kailath, *Linear Sys.*, Englewood Cliffs: Prentice-Hall, 1980.

V. Kucera, *Discrete Linear Control: The Polynomial Equation Approach*, Chichester: John Wiley & Sons, 1979.

A.G.J. MacFarlane and B. Kouvaritakis, "A design technique for linear multivariable feedback systems", *Inter. J. Cont.*, 25, pp. 837–874, 1977.

A.G.J. MacFarlane and I. Postlethwaite, "The generalized Nyquist stability criterion and multivariable root loci", *Inter. J. Cont.*, 25, pp. 81–127, 1977.

M.V. Moreira and J.C. Basilio, "Design of normalizing precompensators via alignment of output-input principal directions", in *44th IEEE Conference on Decision and Control and European Control Conference*, Seville, Spain, 2005, pp. 2170–2175.

C.N. Nett, C.A. Jacobson and M.J. Balas, "A connection between state-space and doubly coprime fractional representations", in *IEEE-Trans. on Autom. Cont.*, Vol. AC-29, 1984, pp. 831–832.

H.H. Rosenbrock, *State space and multivariable theory*, London: Nelson, 1970.

M.C. Smith, "On the generalized Nyquist stability criterion", *Inter. J. Cont.*, 34, pp. 885–920, 1981.

J.H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford: Clarendon Press, 1965.

D.C. Youla, H.A. Jabr and J.J. Bongiorno, "Modern Wiener-Hopf design of optimal controllers – Part II: the multivariable case", *IEEE-Trans. on Autom. Cont.*, 21, pp. 319–338, 1976.