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A normalizing precompensator for the design of effective and reliable commutative controllers

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Sensitivity to parameter perturbation represents the main caution regarding the use of the characteristic locus method on the design of multivariable control systems. The method is not effective when the condition number of the plant eigenvector matrix is high or, equivalently, when the plant transfer matrix differs a great deal from normality. With the view to coping with this problem, it is proposed in this paper a precompensator structure and, in the sequel, two optimization problems are formulated and solved: the first one, for 2×2 systems, aims at minimizing the eigenvector matrix condition number; the second one, for the general $m \times m$ case, is intended to make the precompensated system as normal as possible by minimizing a defined measure of normality. Once the precompensated system matrix has been made close to a normal one, the characteristic locus method can then be applied effectively, leading to reliable control systems, as far as stability in face of uncertainty is concerned.

1. Introduction

The characteristic locus method (CLM), introduced by MacFarlane and Belletruti (1970), is a powerful tool for the design of multivariable linear control systems. It is based on the generalized Nyquist stability criterion (MacFarlane 1970, MacFarlane and Postlethwaite 1977), which extends the Nyquist criterion to multivariable systems. Since Nyquist diagrams combine gain and phase characteristics in one single plot, it provides vital information about relative stability of multivariable systems. However, the CLM has been seriously criticized on account to eigenvalue sensitivity to model perturbation (Doyle and Stein 1981), i.e. it is possible for the characteristic loci of the perturbed system to differ a great deal from those of the nominal system, even for a small perturbation in the parameters of the plant transfer function. The reason for that lies on the fact that the design of commutative controllers is based on the eigenfunctions of the open loop transfer function, and it is well known that the eigenvalues are only upper bounds for the minimum singular value, which is the true indicator of stability robustness. Therefore, the CLM can only prove effective and reliable, from the robustness viewpoint, when the plant transfer function is approximately normal in the necessary frequency range since, in that case, the eigenvalues and singular values are coincidents.

Two attempts have been made to deal with parameter uncertainty within the CLM. The first one (Daniel and Kouvaritakis 1983, 1984) consists in approximating the plant nominal transfer matrix by a normal one. The CLM can then be applied to the normal system obtained after approximation. The second attempt is a more indirect approach and leads to the so called reversed-frame-normalizing-controllers (RFNC) (Hung and MacFarlane 1982, Basilio and Kouvaritakis 1997). Both approaches have the same drawback, namely the possibility of amplification of the radii of the characteristic locus bands.

In this paper we present a precompensation scheme with the view to making the precompensated plant as normal as possible (according to an optimization criterion) in the necessary frequency range. It is worth remarking that, unlike the previous strategies (Hung and MacFarlane 1982, Daniel and Kouvaritakis 1983, 1984, Basilio and Kouvaritakis 1997), the precompensator proposed here does not widen the characteristic locus bands since its infinity norm is made approximately equal to 1.

The paper is structured as follows. Section 2 presents a brief review on the CLM and the effects of plant uncertainties. The problem of normalization of the plant transfer function is tackled in §3. It is initially proposed a precompensator structure for 2×2 systems and, in the sequel, this structure is generalized for the $m \times m$ case. The methodology is illustrated by means of two numerical examples taken from literature in §4: the first one corresponds to a 2×2 system with badly skewed eigenvector (Doyle and Stein 1981) and the second one, a 3×3 system has been taken from Hung and MacFarlane (1982) and represents a linearized model

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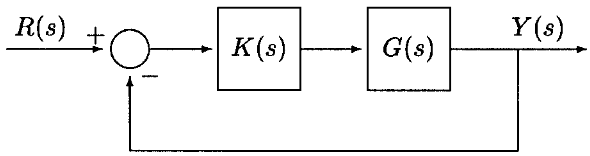

Figure 1. Block diagram of a feedback control system.

of the vertical-plane dynamics of an aircraft. Finally, conclusions are drawn in § 5.

2. Brief review

According to the generalized Nyquist criterion, the feedback system of figure 1 will be stable if and only if the net sum of anticlockwise encirclements of the $-1+j0$ point by the characteristic loci of the open loop transfer function equals the number of unstable poles of the plant and the controller. The characteristic loci of a general transfer function $Q(s)$ are defined as the frequency responses of the eigenfunctions $q(s)$, which are the solutions of the algebraic equation

$$\det[q(s)I - Q(s)] = 0 \tag{1}$$

where $\det(\cdot)$ denotes determinant.

Let us now assume that $G(s)$ and $K(s)$ denote, respectively, the plant and controller transfer functions. In order to apply the CLM to the design of the multivariable control system of figure 1, the first step is to obtain the characteristic value decomposition of $G(s)$. To do so, assume that $d(s)$ denotes the least commom multiple of the denominator polynomials of all entries of $G(s)$. Thus the characteristic value decomposition of $G(s)$ is given by

$$G(s) = \frac{1}{d(s)} N(s) = \frac{1}{d(s)} W(s) \Lambda_N(s) V(s) \tag{2}$$

where $\Lambda_N(s) = \mathrm{diag}\,(n_1(s), n_2(s), \ldots, n_m(s))$ is a diagonal matrix whose diagonal elements are the eigenfunctions of $N(s)$, $W(s) = [\mathbf{w}_1(s) \;\; \mathbf{w}_2(s) \;\; \cdots \;\; \mathbf{w}_n(s)]$ is the eigenvector matrix whose ith column $\mathbf{w}_i(s)$ is the eigenvector function associated to the ith eigenfunction $n_i(s)$ and $V(s) = W^{-1}(s)$ is the dual-eigenvector matrix. In accordance with the CLM, the controller transfer function shares with the plant the same eigenvector and dual-eigenvector matrices, having the form

$$K(s) = W(s) \Lambda_K(s) V(s) \tag{3}$$

where $\Lambda_K(s) = \mathrm{diag}[k_1(s), k_2(s), \ldots, k_m(s)]$, with $k_i(s)$, $i = 1, \ldots, m$, being chosen in such a way that $Q(s) = G(s)K(s)$ satisfies the generalized Nyquist criterion with good gain and phase margins. Note that $K(s)$, defined in (3), commutes with $G(s)$ with respect to multiplication, namely, $G(s)K(s) = K(s)G(s)$, and, for this reason, $K(s)$ has been called a commutative controller.

However, in general, $n_i(s)$ is not a rational function in s and therefore $\mathbf{w}_i(s)$ will not be a rational function as well. This implies that the definition of $K(s)$ given in (3), although theoretically convenient has some practical difficulties, since it may lead to irrational controllers which are difficult to implement. It is necessary therefore to find ways to construct $K(s)$ in order for the resulting controller to be rational. This can be done in the following ways: (i) approximate commutative controllers (ACC) (MacFarlane and Kouvaritakis 1977); (ii) approximately exact commutative controllers (AECC) (Cloud and Kouvaritakis 1987); (iii) causal commutative controllers (CCC) (Kouvaritakis and Basilio 1994) or (iv) rational commutative controllers (RCC) (Basilio and Kouvaritakis 1995).

Since the design of commutative controllers is based on the choice of its eigenfunctions, the CLM can only be effective—from the robust stability point of view—when the plant transfer function is approximately normal (Doyle and Stein 1981). This can be easily seen, for example, with the help of Bauer–Fike's theorem (Bauer and Fike 1960). (Similar conclusions could be drawn if the small gain theorem (Doyle and Stein 1981) were used.) Let us asume additive perturbation to describe the plant model, namely

$$G_P(s) = G(s) + \Delta_G(s) \tag{4}$$

where $\bar{\sigma}[\Delta_G(j\omega)] \le \delta_G(w)$, $\delta_G(w)$ being a non-negative real function of the variable ω and represents an upper bound on the size of perturbation at each frequency. Bauer–Fike's theorem states that at a given frequency ω_0, the eigenvalues of $G_P(j\omega_0)$ are inside discs centred at the eigenvalues of $G(j\omega_0)$ and radii equal $\mathcal{C}[W(j\omega_0)]\delta_G(w_0)$, where $\mathcal{C}(.)$ denotes condition number, namely

$$|g_P(j\omega_0) - g(j\omega_0)| \le \mathcal{C}[W(j\omega_0)]\delta_G(w_0) \tag{5}$$

where $g_P(j\omega_0)$ and $g(j\omega_0)$ are, respectively, the eigenvalues of $G_P(j\omega_0)$ and $G(j\omega_0)$ and $|\cdot|$ denotes the modulus of a complex number. It can be seen from (5) that when $G(j\omega_0)$ is normal, its eigenvector matrix has condition number equal to 1, which implies that the eigenvalues of the perturbed matrix lie inside discs of radii equal to the perturbation magnitude. As the matrix $G(j\omega_0)$ departures from normality, $\mathcal{C}[W(j\omega_0)]$ becomes larger, giving wider regions for the eigenvalues of the perturbed matrix. It is important to remark that Bauer–Fike's theorem gives a condition which is only sufficient, and therefore the upper bound given in (5) is clearly conservative. Despite this, it is possible to see from (5) that when $G(s)$ has skew eigenvectors the characteristic loci of $Q_P(s) = G_P(s)K(s)$ ($K(s)$ designed in accordance with the CLM) and those of $Q(s)$ may differ a great deal even for small values of $\delta_G(\omega)$. The consequence of this fact is that the CLM is not effective for plants with skew eigenvectors.

In dealing with plants with skew eigenvectors in the CLM environment, the following approaches can be followed:

(i) find a normal approximation $(G_N(s))$ for $G(s)$ (Daniel and Kouvaritakis 1983, 1984) and then apply the CLM to $G_N(s)$;

(ii) design RFNCs $K(s)$ (Hung and MacFarlane 1982, Basilio and Kouvaritakis 1997) for $G(s)$;

(iii) find a precompensator $K_P(s)$ which makes the precompensated system $\tilde{G}(s) = G(s)K_P(s)$ as normal as possible and then apply the CLM to $\tilde{G}(s)$.

Approach (i), although effective in some cases, does not solve the CLM sensitivity problem since the radii of the discs containing the eigenvalues of $G_P(s)$ can be as large as before, due to error in approximating a matrix with skew eigenvector by a normal one. RFNCs, as suggested in (ii), indeed makes the compensated system free from sensitivity problems but a dilation on the perturbation magnitude may occur if $\bar{\sigma}[K(j\omega)]$ is made too high. Therefore, the design of a normalizing precompensator (iii) seems to be the best option.

3. A normalizing precompensator

3.1. *Problem formulation*

Let us assume additive perturbation to describe the model as in (4). Therefore after the introduction of a precompensator $K_P(s)$, the model with parameter perturbation of the precompensated plant $\tilde{G}(s) = G(s)K_P(s)$ will be given as

$$\tilde{G}_P(s) = G_P(s)K_P(s) = \tilde{G}(s) + \Delta_{\tilde{G}}(s) \tag{6}$$

where $\bar{\sigma}[\Delta_{\tilde{G}}(j\omega)] \leq \bar{\sigma}[\Delta_G(j\omega)]\bar{\sigma}[K_P(j\omega)]$. Consider the following problem: find a precompensator $K_P(s)$ such that $\tilde{G}(s) = G(s)K_P(s)$ be as normal as possible in the necessary frequency range. It can be seen immediately from (6) that such a precompensator must not only make $\tilde{G}(s) = G(s)K_P(s)$ as normal as possible but also should have the largest singular value approximately equal to 1 in the necessary frequency range in order not to widen the perturbation size.

The problem of finding $K_P(s)$ which makes $\tilde{G}(s)$ as normal as possible can be addressed either by forcing the condition number of the eigenvector matrix of $\tilde{G}(s)$ to be as closely as possible to 1 or by using the definition of normal matrix (a matrix $\tilde{N}$ is said to be normal if and only if it commutes with its associate, i.e. $\tilde{N}\tilde{N}^* = \tilde{N}^*\tilde{N}$). With this definition in mind it is then possible to define measures of normality of a given matrix (see Hung and MacFarlane 1982, p. 41). The minimization of the eigenvector matrix condition number approach can be applied only to 2×2 systems, because in that case it is possible to obtain expressions for the eigenvectors, while the minimization of a measure of normality is suitable for the general $m \times m$ case.

At this stage it is important to remark that, since the CLM is a well established design methodology (MacFarlane and Kouvaritakis 1977, Cloud and Kouvaritakis 1987, Kouvaritakis and Basilio 1994, Basilio and Kouvaritakis 1995), there is no need to be concerned with the controller design for the precompensated system. Therefore, although this paper does not deal with the whole controller design, the results to be presented here will show that in applying the CLM to the design of multivariable control systems the first step is the design of a normalizing precompensator as the one to be proposed in this paper.

3.2. *Normalization by minimization of the condition number of the eigenvector matrix (2 × 2 case)*

Let us write the plant transfer function $G(s)$ as

$$G(s) = \frac{1}{d(s)}N(s) = \frac{1}{d(s)}\begin{bmatrix} n_{11}(s) & n_{12}(s) \\ n_{21}(s) & n_{22}(s) \end{bmatrix} \tag{7}$$

where $d(s)$ denotes the least common multiple of the denominator polynomials of all entries of $G(s)$ and $n_{ij}(s)$, $i,j = 1,2$ are polynomials in s. Assume now that for a given frequency ω, $N(j\omega)$ is

$$N = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} \tag{8}$$

where $n_{ij} \in \mathbb{C}$, $i,j = 1,2$. The problem here is to find a precompensator $(K_P(s))$ that makes $\tilde{G}(j\omega) = G(j\omega)K_P(jw)$ as closely as possible to a normal matrix at each frequency ω. This is equivalent to requiring that the eigenvector matrix of $\tilde{G}(j\omega)$, at each frequency ω, has a condition number as closely as possible to 1. However, since $G(j\omega)$ and $N(jw)$ share the same eigenvectors, the computation of $K_P(j\omega)$ can be performed by considering $N(j\omega)$ instead of $G(j\omega)$. For 2×2 systems it is not difficult to derive an expression for the eigenvector matrix of $N(j\omega)$ which depends solely on the elements of N, as

$$W = \begin{bmatrix} \dfrac{n_{11} - n_{22} + \sqrt{\Delta}}{a} & \dfrac{n_{11} - n_{22} - \sqrt{\Delta}}{b} \\ \dfrac{2n_{21}}{a} & \dfrac{2n_{21}}{b} \end{bmatrix} \tag{9}$$

where

$$\left.\begin{aligned} a^2 &= |n_{11} - n_{22} + \sqrt{\Delta}|^2 + 4|n_{21}|^2 \\ b^2 &= |n_{11} - n_{22} - \sqrt{\Delta}|^2 + 4|n_{21}|^2 \end{aligned}\right\} \tag{10}$$

and

$$\Delta = (n_{11} - n_{22})^2 + 4n_{12}n_{21} \tag{11}$$

Note that the column vectors of W have all unity euclidean norm. This is necessary since what is intended here is to use the condition number of W as an indicator of the normality of N. If the eigenvectors do not have unity euclidean norm, W may have a huge condition number even in the particular case when its column vectors are approximately orthogonal.

Let us now define the following structure for the precompensator

$$K_P(j\omega) = \begin{bmatrix} 0 & 1 \\ r(j\omega)\,e^{j\theta(j\omega)} & 0 \end{bmatrix} \tag{12}$$

where $\theta(j\omega) \in [0, 2\pi)$ and $r(j\omega) \in (0, 1]$, $\theta(j\omega)$ and $r(j\omega)$ are computed in order to make $\tilde{G}(j\omega) = G(j\omega)K_P(j\omega)$ as near as possible to a normal matrix at each frequency ω.

The motivation for structure (12) comes from the static precompensator

$$K_H = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{13}$$

which is obtained when the ALIGN algorithm (MacFarlane and Kouvaritakis 1977) is deployed to design a precompensator with the view to reducing interaction at high frequencies. The effect on $G(s)$ of such a precompensator is that $G(s)K_H$ has eigenvectors which are nearly aligned with the standard basis vectors at high frequencies (the frequencies where the algorithm is employed), therefore reducing the condition number of the eigenvector matrix of $G(s)K_H$ at those frequencies. However, at low frequencies, the condition number of the eigenvector matrix, in general, increases. With the view to overcome this limitations we propose the precompensator structure (12) which is dynamic and consequently has more degrees of freedom.

The normalization problem can then be stated as follows: for a given frequency ω, compute r and θ which minimizes the condition number of the eigenvector matrix ($\tilde{W}$) of $\tilde{N} = NK_P$. Remember that the computation of the condition numbers of W and $\tilde{W}$ requires the knowledge of their maximum and minimum singular values, which are, respectively, the largest and the smallest roots of the equation

$$\lambda^2 - \mathrm{tr}\,(W^*W)\lambda + \det\,(W^*W) = 0 \tag{14}$$

where $\mathrm{tr}\,(.)$ denotes trace of a matrix. After some straightforward manipulation we obtain

$$\lambda_{1,2} = ab \pm \sqrt{a^2b^2 - 16|n_{21}|^2|\Delta|} \tag{15}$$

where a and b are given in (10). Therefore the condition number of W, $\mathcal{C}^2(W)$, can be written as

$$\mathcal{C}^2(W) = \frac{ab + \sqrt{a^2b^2 - 16|n_{21}|^2|\Delta|}}{ab - \sqrt{a^2b^2 - 16|n_{21}|^2|\Delta|}} \tag{16}$$

Defining

$$X = |n_{11} - n_{22}|^2 + 2(|n_{12}|^2 + |n_{21}|^2) \tag{17}$$

it is possible to re-write (16) as

$$\mathcal{C}^2(W) = \frac{\sqrt{X + |\Delta|} + \sqrt{X - |\Delta|}}{\sqrt{X + |\Delta|} - \sqrt{X - |\Delta|}} \tag{18}$$

Once an expression for the square of the condition number of W has been derived, the next step is to obtain a similar expression for the eigenvector matrix of $\tilde{G} = GK_P$, which can be given as

$$\tilde{G} = GK_P = \frac{1}{d}NK_P = \frac{1}{d}\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ r\,e^{j\theta} & 0 \end{bmatrix} \tag{19}$$

and writing $\tilde{G} = (1/d)\tilde{N}$, we obtain

$$\tilde{N} = \begin{bmatrix} \gamma n_{12} & n_{11} \\ \gamma n_{22} & n_{21} \end{bmatrix} \tag{20}$$

where $\gamma = r\,e^{j\theta}$. In expressions (8)–(20), the dependence on the frequency ω has been omitted, but it is important to remark that the values of r and θ are frequency dependent. Note that $\tilde{N}$ has the same structure as N and, consequently its eigenvector matrix ($\tilde{W}$) has a similar form to W, given in (9), but with n_{11}, n_{12}, n_{21} and n_{22} replaced, respectively, by $\tilde{n}_{11} = \gamma n_{12}$, $\tilde{n}_{12} = n_{11}$, $\tilde{n}_{21} = \gamma n_{22}$ and $\tilde{n}_{22} = n_{21}$. Thus the following expression for the square of the condition number of $\tilde{W}$ can be immediately written

$$\mathcal{C}^2(\tilde{W}) = \frac{\sqrt{\tilde{X} + |\tilde{\Delta}|} + \sqrt{\tilde{X} - |\tilde{\Delta}|}}{\sqrt{\tilde{X} + |\tilde{\Delta}|} - \sqrt{\tilde{X} - |\tilde{\Delta}|}} \tag{21}$$

where

$$\tilde{X} = |\gamma n_{12} - n_{21}|^2 + 2(|n_{11}|^2 + |\gamma n_{22}|^2) \tag{22}$$

$$\tilde{\Delta} = (\gamma n_{12} - n_{21})^2 + 4\gamma n_{11}n_{22} \tag{23}$$

The problem of designing K_P requires then the computation of r ($0 < r \leq 1$) and θ ($0 \leq \theta < 2\pi$) for which the matrix $\tilde{G} = GK_P$ has an eigenvector matrix whose condition number is as closely as possible to one. Such a pair (r, θ) does not necessarily exist, as shown in the following lemma.

Lemma 1: *The condition number of $\tilde{W}$ is smaller than that of W, i.e. $\mathcal{C}(\tilde{W}) < \mathcal{C}(W)$, if and only if for a given pair (r, θ), $\rho(r, \theta) < 1$, where*

$$\rho(r, \theta) = \frac{\tilde{X}(r, \theta)\,|\Delta|}{|\tilde{\Delta}(r, \theta)|\,X} \tag{24}$$

Proof: Since the condition number of a matrix is always greater or equal 1, we may write

$$\mathcal{C}(\tilde{W}) < \mathcal{C}(W) \iff \mathcal{C}^2(\tilde{W}) < \mathcal{C}^2(W) \tag{25}$$

and substituting (18) and (21) in (25), we obtain

$$\frac{\sqrt{\tilde{X}+|\tilde{\Delta}|}+\sqrt{\tilde{X}-|\tilde{\Delta}|}}{\sqrt{\tilde{X}+|\tilde{\Delta}|}-\sqrt{\tilde{X}-|\tilde{\Delta}|}} < \frac{\sqrt{X+|\Delta|}+\sqrt{X-|\Delta|}}{\sqrt{X+|\Delta|}-\sqrt{X-|\Delta|}} \tag{26}$$

After some straightforward calculation, inequality (26) can be written as

$$\sqrt{\tilde{X}-|\tilde{\Delta}|}\sqrt{X+|\Delta|}-\sqrt{\tilde{X}+|\tilde{\Delta}|}\sqrt{X-|\Delta|} < 0 \tag{27}$$

or, equivalently

$$\frac{\tilde{X}|\Delta|}{X|\tilde{\Delta}|} < 1 \tag{28}$$

Finally, defining $\rho = \tilde{X}|\Delta|/(X|\tilde{\Delta}|)$, completes the proof of the lemma. □

Lemma 1 provides more than a simple test to check if a pair $(r,\theta) \in \mathcal{R}$, where $\mathcal{R} = (0,1] \times [0,2\pi)$ makes the matrix $\tilde{G}$ closer than G to a normal matrix. As we are going to see in the sequel, the problem of finding K_P that makes the condition number of $\tilde{W}$ smaller than that of W turns out to be the one of finding $(r,\theta) \in \mathcal{R}$ that minimizes $\rho(r,\theta)$. In order to do so, let us first define

$$\alpha = \frac{|\Delta|}{X} \quad \text{and} \quad \tilde{\alpha}(r,\theta) = \frac{|\tilde{\Delta}(r,\theta)|}{\tilde{\mathrm{X}}(r,\theta)} \tag{29}$$

We may then state the following result.

Lemma 2: *W has infinite condition number if and only if α is approximately zero.*

Proof: Substituting α, given in (29), in (18) we obtain

$$\mathcal{C}^2(W) = \frac{1+\sqrt{1-\alpha^2}}{\alpha} \tag{30}$$

$(\Longrightarrow)$ $\alpha \to 0 \Longrightarrow \mathcal{C}^2(W) \to +\infty$.
$(\Longleftarrow)$ For $\alpha \neq 0$, equation (30) may be written as

$$\alpha[(\mathcal{C}^4(W)+1)\alpha - 2\mathcal{C}^2(W)] = 0 \tag{31}$$

Thus

$$\alpha = \frac{2\mathcal{C}^2(W)}{1+\mathcal{C}^4(W)} \tag{32}$$

is the unique solution to (31). It is immediate to see that $\alpha \to 0$ when $\mathcal{C}(W) \to +\infty$. □

The result presented in Lemma 2 can be interpreted with the help of (29) as follows: in order for α to approach zero either X has to go to infinity when $|\Delta|$ is kept finite, or the opposite, i.e. for X finite, $|\Delta|$ is made arbitrarily small. In order for X to approach zero, the following conditions should be met: $n_{11} \approx n_{22}$, $n_{12} \to 0$ and $n_{21} \to 0$. However if this happens then Δ also approaches zero since N becomes a diagonal matrix. It is not hard to check that in this case $\alpha \to 1$, which implies that α cannot approach zero when $X \to 0$. On the other hand, when $|\Delta(j\omega)|$ approaches zero, the eigenvalues become approximately equal which ultimately implies that the eigenvectors are getting nearly parallel, giving rise to a nearly singular eigenvector matrix, or equivalently, a matrix with a huge condition number. This is the worst case for the application of the CLM since the resulting compensated system will be very sensitive to parameter perturbation at that frequency, as predicted by Bauer–Fike's theorem. However, as the following result shows, the use of precompensator (12) can actually turn the precompensated system into one whose eigenvector matrix condition number is approximately 1, which represents the best starting point for the design of commutative controllers, as far as sensitivity to parameter perturbation is concerned.

Theorem 1: *Let N be such that its eigenvector matrix W has infinite condition number, i.e., $\mathcal{C}(W) \to +\infty$. If for a pair $(r,\theta) \in \mathcal{R}$, $\rho(r,\theta) \to 0$ then the condition number of $\tilde{W}$ will be approximately equal to 1, i.e. $\mathcal{C}(\tilde{W}) \to 1$.*

Proof: Substituting (29) in (21) we obtain

$$\mathcal{C}^2(\tilde{W}) = \frac{1+\sqrt{1-\tilde{\alpha}^2(r,\theta)}}{\tilde{\alpha}(r,\theta)} \tag{33}$$

and from equations (30) and (33) we may conclude that

$$0 < \alpha \le 1 \quad \text{and} \quad 0 < \tilde{\alpha}(r,\theta) \le 1 \tag{34}$$

Since $\tilde{\alpha}(r,\theta)$ is always smaller or equal to 1, then

$$\rho(r,\theta) = \frac{\alpha}{\tilde{\alpha}} \ge \alpha \tag{35}$$

From lemma 2, we know that $\mathcal{C}(W) \to +\infty$ if and only if $\alpha \to 0$. Therefore, if there exists a pair $(r,\theta) \in \mathcal{R}$ such that $\rho(r,\theta) \to 0$, then by (35), $\rho(r,\theta) \to \alpha$ or equivalently, $\tilde{\alpha}(r,\theta) \to 1$. Finally, the result of the theorem is obtained by taking the limit of $\mathcal{C}^2(\tilde{W})$ when $\tilde{\alpha}$ approaches 1. □

Theorem 1 shows that in order to make, at a frequency ω, the condition number of the eigenvector matrix of the precompensated system as closely as possible to 1, it is necessary to find a pair (r,θ) which minimizes $\rho(r,\theta)$, or equivalently, maximizes $\tilde{\alpha}(r,\theta)$. This leads to the following optimization problem.

Problem 1: $\max_{(r,\theta)\in\mathcal{R}} \tilde{\alpha}(r,\theta)$, where, according to equations (22), (23) and (29), $\tilde{\alpha}(r,\theta)$ is given as

$$\tilde{\alpha}(r,\theta) = \frac{|(r\,e^{j\theta}n_{12} - n_{21})^2 + 4\gamma n_{11}n_{22}|}{|r\,e^{j\theta}n_{12} - n_{21}|^2 + 2(|n_{11}|^2 + r^2|n_{22}|^2)} \tag{36}$$

We can now derive an algorithm for the computation of a precompensator $K_P(s)$ such that the eigenvector matrix of $\tilde{G}(s)$ has a condition number which is smaller or equal to that of the eigenvector matrix of $G(s)$ and is as closely as possible to 1 in all the necessary frequency ranges.

Algorithm 1:

Step 1. Select a finite number of frequencies ω_k, $k = 0, 1, \ldots, q$.

Step 2. For each frequency ω_k, $k = 0, 1, \ldots, q$ compute $N(j\omega_k)$, $\alpha(j\omega_k)$ and employing any numerical optimization method find a pair $(r(j\omega_k), \theta(j\omega_k)) \in \mathcal{R}$ that maximizes $\tilde{\alpha}[r(j\omega_k), \theta(j\omega_k)]$.

Step 3. Compute

$$\rho(r(j\omega_k), \theta(j\omega_k)) = \alpha(j\omega_k)/\tilde{\alpha}[(j\omega_k), \theta(j\omega_k)] \tag{37}$$

Step 4. If $\rho(r(j\omega_k), \theta(j\omega_k)) \geq 1$ then

$$K_P(j\omega_k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{38}$$

else, i.e. if $\rho(r(j\omega_k), \theta(j\omega_k)) < 1$, then

$$K_P(j\omega_k) = \begin{bmatrix} 0 & 1 \\ r(j\omega_k)\,e^{j\theta(j\omega_k)} & 0 \end{bmatrix} \tag{39}$$

Step 5. Obtain stable and minimum phase transfer functions for each entry of $K_P(s)$ in such a way that the frequency responses of its elements approximately match those obtained in Step 4 and are smooth at the discontinuity points of the frequency responses obtained in Step 4.

3.3. *Normalization by minimization of the deviation from normality*

3.3.1. *Generalization of the precompensator structure for the $m \times m$ case.* The precompensator structure (12) for the 2×2 case is derived as follows: (1) start from the 2×2 identity matrix; (2) multiply the second column by $r\,e^{j\theta}$; (3) swap the columns of the resulting matrix. A generalization of this structure for the general $m \times m$ case can be carried out as follows: (1) start from the identity matrix of order m; (2) multiply column l by $r(j\omega)\,e^{j\theta(j\omega)}$; (3) swap columns k and l of the matrix obtained in step (2). Denoting $K_{P_{kl}}(j\omega)$ the matrix generated after step (3), we have

$$K_{P_{kl}}(j\omega) = [\mathbf{e}_1 \;\ldots\; \mathbf{e}_{k-1} \;\; r(j\omega)\,e^{j\theta(j\omega)}\mathbf{e}_l \;\; \mathbf{e}_{k+1} \;\ldots\; \mathbf{e}_{l-1} \;\; \mathbf{e}_k \;\; \mathbf{e}_{l+1} \;\ldots\; \mathbf{e}_m] \tag{40}$$

where the columns $\mathbf{e}_1 \ldots \mathbf{e}_{k-1}$, $\mathbf{e}_{k+1} \ldots \mathbf{e}_{l-1}$ and $\mathbf{e}_{l+1} \ldots \mathbf{e}_m$ will appear in $K_{P_{kl}}(j\omega)$ only when $k \geq 2$, $l \geq k+2$ or $l \leq m-1$, respectively. Note that, in the general $m \times m$ case, there are several ways to obtain $K_{P_{kl}}(j\omega)$, depending on the column that is multiplied by $r\,e^{j\theta}$ and on those which are swapped. It is well known from the combinatorial analysis that if a set has m elements, then the total number of its subsets consisting of p elements each is equal

$$\binom{m}{p} = \frac{m!}{p!(m-p)!} \tag{41}$$

Therefore the number of structures (similar to (12)) for the general case is $\binom{m}{2}$.

3.3.2. *A measure of normality.* Let us now define a measure of how close to normality a given matrix is. It is well known that a matrix $\tilde{G}: m \times m$ is normal if and only if it commutes with its associate $\tilde{G}^*$, i.e. if and only if $\tilde{G}^*\tilde{G} = \tilde{G}\tilde{G}^*$. Therefore the following expression can be used to measure the deviation of a matrix from normality

$$\delta(\tilde{G}) = \frac{\|\tilde{E}\|_{\mathcal{F}}^2}{\|\tilde{G}^*\tilde{G}\|_{\mathcal{F}}^2} = \frac{\|\tilde{E}\|_{\mathcal{F}}^2}{\|\tilde{G}\tilde{G}^*\|_{\mathcal{F}}^2} \tag{42}$$

where $\tilde{E} = \tilde{G}^*\tilde{G} - \tilde{G}\tilde{G}^*$ and $\|\tilde{E}\|_{\mathcal{F}}$ denotes the Frobenius norm of $\tilde{E}$, which is defined as

$$\|\tilde{E}\|_{\mathcal{F}}^2 = \sum_{i,j=1}^{m} |\tilde{e}_{ij}|^2 = \operatorname{tr}(\tilde{E}^*\tilde{E}) \tag{43}$$

with e_{ij} denoting the (i,j) element of $\tilde{E}$. Note that the closer to 0 $\delta(\tilde{G})$ is the nearer to a normal matrix $\tilde{G}$ will be. It is also important to remark on the use of $\|\tilde{G}^*\tilde{G}\|_{\mathcal{F}}^2$ or $\|\tilde{G}\tilde{G}^*\|_{\mathcal{F}}^2$ in the denominator of (42) in order to account for the size of the elements of $\tilde{G}$.

3.3.3. *A minimization problem.* Let $\mathcal{C}_2$ denote the set whose elements are the pairs (k,l) formed with all the combinations of the elements of $\{1, 2, \ldots, m\}$ taken two at a time (m denoting the order of $G(s)$). The problem of finding a precompensator $K_{P_{kl}}(j\omega)$ which makes the matrix $\tilde{G}_{kl}(j\omega) = G(j\omega)K_{P_{kl}}(j\omega)$ as closely as possible to a normal matrix, according to the measure defined in (42), can be stated as follows:

Problem 2: $\min_{(k,l)\in\mathcal{C}_2} \min_{(r(j\omega),\theta(j\omega))\in\mathcal{R}} J_{kl}(r,\theta)$, for each ω in the necessary frequency range, where

$$J_{kl}(r,\theta) = \frac{\|\tilde{E}_{kl}\|_{\mathcal{F}}^2}{\|\tilde{G}_{kl}^*\tilde{G}_{kl}\|_{\mathcal{F}}^2} \tag{44}$$

and $\tilde{E}_{kl}(j\omega) = \tilde{G}_{kl}^*\tilde{G}_{kl} - \tilde{G}_{kl}\tilde{G}_{kl}^*$.

Using trace properties, it is possible to re-write (44) as

$$\tfrac{1}{2}J_{kl}(r,\theta) = 1 - \frac{\mathrm{tr}\,(\tilde{G}_{kl}^{*}\tilde{G}_{kl}\tilde{G}_{kl}\tilde{G}_{kl}^{*})}{\mathrm{tr}\,[(\tilde{G}_{kl}^{*}\tilde{G}_{kl})^{2}]} = 1 - \tilde{J}_{kl}(r,\theta) \quad (45)$$

where

$$\tilde{J}_{kl}(r,\theta) = \frac{\mathrm{tr}\,(\tilde{G}_{kl}^{*}\tilde{G}_{kl}\tilde{G}_{kl}\tilde{G}_{kl}^{*})}{\mathrm{tr}\,[(\tilde{G}_{kl}^{*}\tilde{G}_{kl})^{2}]} \quad (46)$$

and since $J_{kl}(r,\theta) \geq 0$ then $\tilde{J}_{kl}(r,\theta) \leq 1$. Thus, Problem 2 is equivalent to:

Problem 3: $\max_{(k,l)\in\mathcal{C}_2} \max_{(r,\theta)\in\mathcal{R}} \tilde{J}_{kl}(r,\theta)$, for each ω in the frequency range. □

The first step towards the solution of Problem 3 is to obtain an expression for $\tilde{J}_{kl}$ that depends uniquely on r and θ. Such an expression may be derived if we note that $\tilde{G}_{kl} = GK_{P_{kl}}$ can be written as

$$\tilde{G}_{kl} = P_{kl} + Q_{kl} r\, \mathrm{e}^{j\theta} \quad (47)$$

where P_{kl} is a matrix whose columns are the same as those of G, except columns k, which is identically zero and column l which is identical to column k of G and Q_{kl} is a matrix whose unique non-zero column is column k, which is equal to column l of G, as

$$P_{kl} = [\,\mathbf{g}_1 \quad \cdots \quad \mathbf{g}_{k-1} \quad \mathbf{0} \quad \mathbf{g}_{k+1} \quad \cdots \quad \mathbf{g}_{l-1} \quad \mathbf{g}_k \quad \mathbf{g}_{l+1} \quad \cdots \quad \mathbf{g}_m\,] \quad (48)$$

$$Q_{kl} = \mathbf{g}_l \mathbf{e}_k^{t} = \Big[\underbrace{\mathbf{0} \ \ \cdots \ \ \mathbf{0}}_{k-1} \quad \mathbf{g}_l \quad \underbrace{\mathbf{0} \ \ \cdots \ \ \mathbf{0}}_{m-k}\Big] \quad (49)$$

Substituting $\tilde{G}_{kl}$, as given by (47), in (46) and using the easily verified fact that $Q_{kl}P_{kl}^{*} = 0$, then equation (46) turns out to be

$$\tilde{J}_{kl}(r,\theta) = \frac{b(r,\theta)}{a(r)} = \frac{b_0 r^4 + b_1(\theta) r^3 + b_2 r^2 + b_3(\theta) r + b_4}{a_0 r^4 + a_2 r^2 + a_4} \quad (50)$$

where

$$b_0 = \mathrm{tr}\,(Q^{*}QQQ^{*}) = \|\mathbf{g}_l\|_2^2 |g_{kl}|^2 \quad (51)$$

$$b_1(\theta) = v_1 \cos\theta - u_1 \sin\theta \quad (52)$$

$$\begin{aligned} b_2 &= \mathrm{tr}\,(P^{*}PQQ^{*}) + \mathrm{tr}\,(PP^{*}Q^{*}Q) \\ &= (\|\mathbf{z}_{kl}\|_2^2 + \|\mathbf{g}_l\|_2^2 \|\mathbf{p}_k^{t}\|_2^2) \end{aligned} \quad (53)$$

$$b_3(\theta) = v_3 \cos\theta - u_3 \sin\theta \quad (54)$$

$$b_4 = \mathrm{tr}\,(P^{*}PPP^{*}) \quad (55)$$

$$a_0 = \mathrm{tr}\,[(Q^{*}Q)^2] = \|\mathbf{g}_l\|_2^4 \quad (56)$$

$$a_2 = 2\sum_{i=1, i\neq l}^{m} |\mathbf{g}_l^{*}\mathbf{g}_i|^2 \quad (57)$$

$$a_4 = \mathrm{tr}\,[(P^{*}P)^2] \quad (58)$$

$$u_1 = 2\,\mathrm{Im}\,[\bar{g}_{kl}\mathbf{g}_l^{*}P^{*}\mathbf{g}_l] \quad (59)$$

$$v_1 = 2\,\mathrm{Re}\,[\bar{g}_{kl}\mathbf{g}_l^{*}P^{*}\mathbf{g}_l] \quad (60)$$

$$\mathbf{z}_{kl} = \sum_{i=1, i\neq k,l}^{m} g_{il}\mathbf{g}_i + g_{ll}\mathbf{g}_k \quad (61)$$

$$u_3 = 2\,\mathrm{Im}\left[\mathbf{g}_k^{*}\mathbf{g}_l \sum_{i=1, i\neq l}^{m} g_{ki}\bar{g}_{li} + \sum_{j=1, j\neq k,l}^{m} \mathbf{g}_j^{*}\mathbf{g}_l \sum_{i=1, i\neq l}^{m} g_{ki}\bar{g}_{ji}\right] \quad (62)$$

$$v_3 = 2\,\mathrm{Re}\left[\mathbf{g}_k^{*}\mathbf{g}_l \sum_{i=1, i\neq l}^{m} g_{ki}\bar{g}_{li} + \sum_{j=1, j\neq k,l}^{m} \mathbf{g}_j^{*}\mathbf{g}_l \sum_{i=1, i\neq l}^{m} g_{ki}\bar{g}_{ji}\right] \quad (63)$$

In the formulae above $\mathbf{p}_k^{t}$ denotes the kth row of P, $\mathbf{g}_i$, $i = 1, \ldots, m$ denotes the ith column of G, g_{ij} stands for element (i,j) of G and $\bar{g}_{ij}$ stands for the complex conjugate of g_{ij}. Minimization problem 3 can then be solved in accordance with the following theorem.

Theorem 2: *The maximum value of $\tilde{J}_{k,l}(r,\theta)$ is always achieved by a pair (r,θ) belonging to the non-empty set $\mathcal{P}$, which is given by*

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \{(1,0)\}$$

where

$\mathcal{P}_1 = \{(1,\theta_1), (1, \theta_1 + \pi)$:

$\theta_1 = \arctan\,(-(u_1+u_3)/(v_1+v_3)\}$

$\mathcal{P}_2 = \{(r_i, 0)$: *$r_i$ is a solution of* $a_0 v_1 r^6 + 2(a_0 b_2 - a_2 b_0) r^5$

$+ (3a_0 v_3 - a_2 v_1) r^4 + 4(a_0 b_4 - a_4 b_0) r^3$

$+ (a_2 v_3 - 3a_4 v_1) r^2 + 2(a_2 b_4 - a_4 b_2) r - a_4 v_3 = 0$

and $0 < r_i < 1\}$

$\mathcal{P}_3 = \{(r_{ji}, \theta_i)$, *for* $i = 1, 2$: *r_{ji} is a solution of* $\beta_1 r_{ji}^5$

$+ \beta_2(\theta_i) r_{ji}^4 + \beta_3 r_{ji}^3 + \beta_4(\theta_i) r_{ji}^2 + \beta_5 r_{ji} + \beta_6(\theta_i) = 0$

and $0 < r_{ji} < 1$, *with* $\theta_1 = \arctan\,(-(u_3/v_3))$

and $\theta_2 = \theta_1 - \pi\}$, *when* $u_v = v_1 = 0$ *and either*

$v_3 \neq 0$ *or* $u_3 \neq 0$;

$$= \{(r_{ji},\theta_i),\, for\ i=1,2{:}\ r_{ji}\ is\ a\ solution\ of\ \beta_0(\theta_i)r_{ji}^5 + \beta_1 r_{ji}^4 + \beta_2(\theta_i)r_{ji}^3 + \beta_3(\theta_i)r_{ji}^2 + \beta_4(\theta_i)r_{ji} + \beta_5 = 0,$$
$$with\ \theta_1 = \arctan(-(u_1/v_1)),\theta_2 = \theta_1 - \pi\},$$
$$when\ u_3 = v_3 = 0\ and\ either\ v_1 \neq 0\ or\ u_1 \neq 0$$
$$= \{(r_{1i},\theta_i),(r_{1i},\theta_i - \pi),(r_{2i},\pi - \theta_i)\ and\ (r_{2i},\theta_i),$$
$$where\ \theta_i = \arccos x_i = \arcsin y_i,\ x_i = \sqrt{1-t_i}$$
$$and\ y_i = \sqrt{t_i},t_i\ is\ a\ solution\ of\ \tau_0 t^8 + \tau_1 t^7 + \tau_2 t^6 + \tau_3 t^5 + \tau_4 t^4 + \tau_5 t^3 + \tau_6 t^2 + \tau_7 t + \tau_8 = 0\ in\ the$$
$$interval\ (0,1]\ and\ r_{1i} = r(x_i,y_i),r_{2i} = r(-x_i,y_i),$$
$$with\ r(x,y) = \sqrt{-(u_3x + v_3y)/(u_1x + v_1y)}\},$$
$$when\ either\ v_1 \neq 0\ or\ u_1 \neq 0\ and\ either\ v_3 \neq 0\ or\ u_3 \neq 0;$$
$$= \varnothing,\ when\ u_1 = v_1 = u_3 = v_3$$

where expressions for $\beta_0,\ldots,\beta_6$ and $\tau_0,\ldots,\tau_8$ are given in the Appendix.

Proof: See the Appendix.

Remark: It is important to stress that Theorem 2 above not only shows that the optimization problem $\max_{(r,\theta)\in\mathcal{R}} \tilde{J}_{k,l}(r,\theta)$ always has a solution but also provides the means to compute it. In addition note that since the pairs (r,θ) which form the set $\mathcal{P}_2$ come from the solution of a six degree polynomial equation, and the elements of $\mathcal{P}_3$ are solutions of a polynomial equation of degree not higher than eight then it is straightforward to see that $\mathcal{P}$ has at most 17 elements. Therefore the pair (r,θ) for which $\tilde{J}_{k,l}(r,\theta)$ attains its maximum can be obtained by direct evaluation of (50) for all elements of $\mathcal{P}$.

Based on Theorem 2 and on the definition of $K_{P_{kl}}(s)$ we may derive the following algorithm for the computation of $K_P(s)$ which makes $\tilde{G}(s) = G(s)K_P(s)$ as closely as possible to a normal matrix according to measure (42).

Algorithm 2:

Step 1. Select a finite number of frequency points ω_i, $i = 0,1,\ldots,q$ in the necessary frequency range.

Step 2. For each frequency ω_i, $i = 0,1,\ldots,q$, compute $G(j\omega_i)$.

Step 3. Form $\binom{m}{2}$ possible precompensators $K_{P_{kl}}(j\omega_i)$ and for each one find the pair $(r_{kl}(j\omega_i), \theta_{kl}(j\omega_i))$ which maximizes $\tilde{J}_{kl}(r_{kl}(j\omega_i),\theta_{kl}(j\omega_i))$ in accordance with Theorem 2. After that, choose the pair (k,l) which gives the smallest value for $J_{kl}(r_{kl}(j\omega_i),\theta_{kl}(j\omega_i))$. Let J_{kl} be such value.

Step 4. Compute

$$J_I = \frac{\|E(j\omega_i)\|_{\mathcal{F}}^2}{\|G^*(j\omega_i)G(j\omega_i)\|_{\mathcal{F}}^2} \tag{64}$$

where

$$E(j\omega_i) = G^*(j\omega_i)G(j\omega_i) - G(j\omega_i)G^*(j\omega_i)$$

Step 5. If $J_I < J_{kl}$ then

$$K_P(j\omega_i) = I_m \tag{65}$$

where I_m denotes the identity matrix of order m. Else, i.e. if $J_I \geq J_{kl}$, then

$$K_P(j\omega_i) = K_{P_{kl}}(j\omega_i) = [\mathbf{e}_1\ \ldots\ \mathbf{e}_{k-1}\ \ r_{kl}(j\omega_i)e^{j\theta_{kl}(j\omega_i)}\mathbf{e}_l\ \ \mathbf{e}_{k+1}\ \ldots\ \mathbf{e}_{l-1}\ \ \mathbf{e}_k\ \ \mathbf{e}_{l+1}\ \ldots\ \mathbf{e}_m] \tag{66}$$

Step 6. Find stable and minimum phase transfer functions for each entry of $K_P(s)$ in such a way that the frequency response of each element is smooth at the discontinuity points of the frequency responses obtained in step 5.

4. Examples

In this section the precompensation scheme presented in the paper is illustrated by means of two numerical examples: the first one represents the nominal transfer function of a 2×2 system and was introduced by Doyle and Stein (1981) in order to bring to light the CLM sensitivity problems; the second one (Hung and MacFarlane 1982) corresponds to a 3×3 system and represents a linearized model of the vertical plane dynamics of an aircraft.

4.1. *Example 1*

Let

$$G(s) = \frac{1}{d(s)}N(s) \tag{67}$$

where $d(s) = (s+1)(s+2)$ and

$$N(s) = \begin{bmatrix} -47s+2 & 56s \\ -42s & 50s+2 \end{bmatrix} \tag{68}$$

The main feature of this plant is the huge condition number of the eigenvector matrix (approximately 196) for all frequencies except at DC, as shown in figure 2(*a*). Note that for $\omega = 0$ then $G(0) = 2I_2$ (I_2 denoting the identity matrix of order 2) which represents a normal matrix. Hence if the condition number of $W(0)$ had been plotted in the same graph there would be a jump from 1 to 196 at the neighbourhood of $\omega = 0$. Similar conclusions can be drawn by considering the measure of

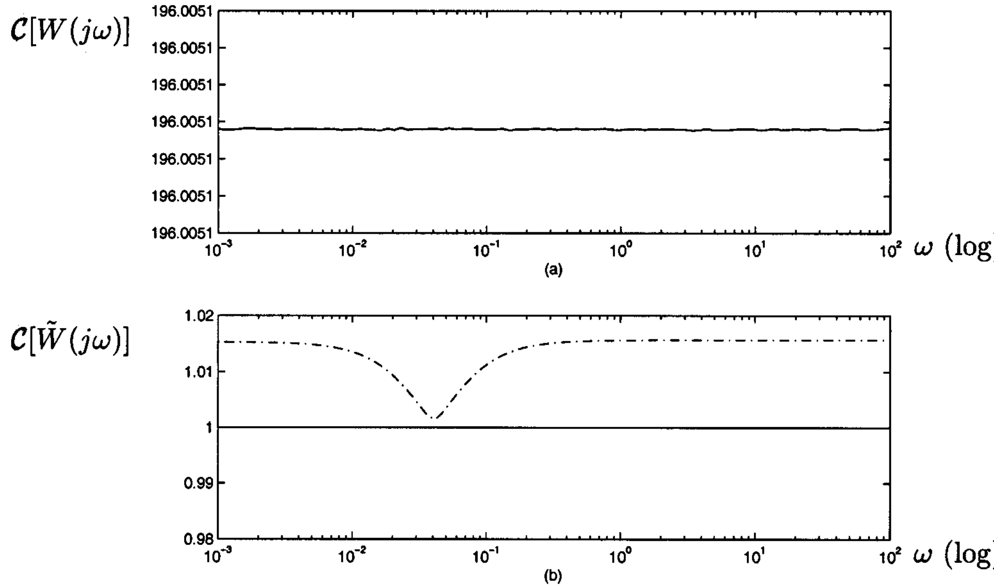

Figure 2. (*a*) Condition number of the eigenvector matrix of $N(j\omega)$; (b) Condition number of the eigenvector matrix of $\tilde{N}(j\omega)$ for $r(j\omega)$ and $\theta(j\omega)$ (—) and $k_{p_{21}}(j\omega) = -0.97$ (– · –).

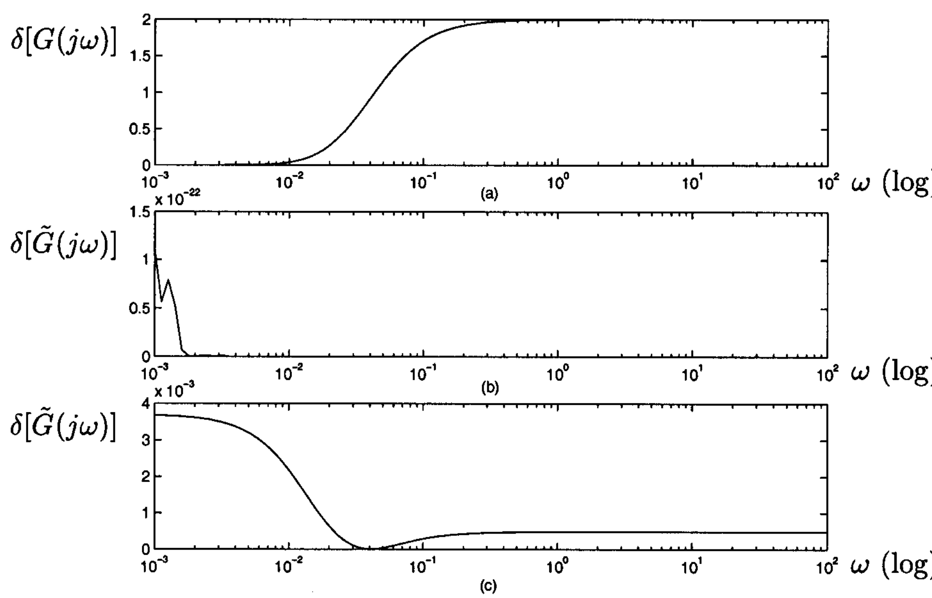

Figure 3. (*a*) $\delta[G(j\omega)]$; (*b*) $\delta[\tilde{G}(j\omega)]$ for $k_{p_{21}}(j\omega) = \gamma(j\omega)$; (*c*) $k_{p_{21}}(s) = -0.97$.

normality (δ) defined in equation (42). It can be seen from figure 3(*a*) that $\delta(\omega)$ is approximately zero for frequencies near 0 and increases to 2 as ω goes to infinity. This behaviour can be explained by noting that $\delta(\omega)$ is a continuous function of ω and thus cannot change abruptly. These facts suggest that in order to draw any conclusion about the normality of a matrix the two measures should be considered together since the condition number shows the normality of the matrix for those frequencies represented in the graph whereas the measure $\delta[G(j\omega)]$ provides information on the normality at DC, at infinity and also at the frequency points where there are jumps on the condition number of the eigenvector matrix. Therefore, from figures 2(*a*) and 3(*a*), we may conclude that $G(j\omega)$ is far from normal for all $\omega \neq 0$, which implies that the CLM cannot be applied directly to $G(s)$, in which case the system would be extremely sensitive to parameter perturbation, as shown in Doyle and Stein (1981). Thus normalization is the unique way to overcome this problem. Notice that, according to Algorithms 1 and 2, the precompensator, at each frequency w, will either be equal to the identity

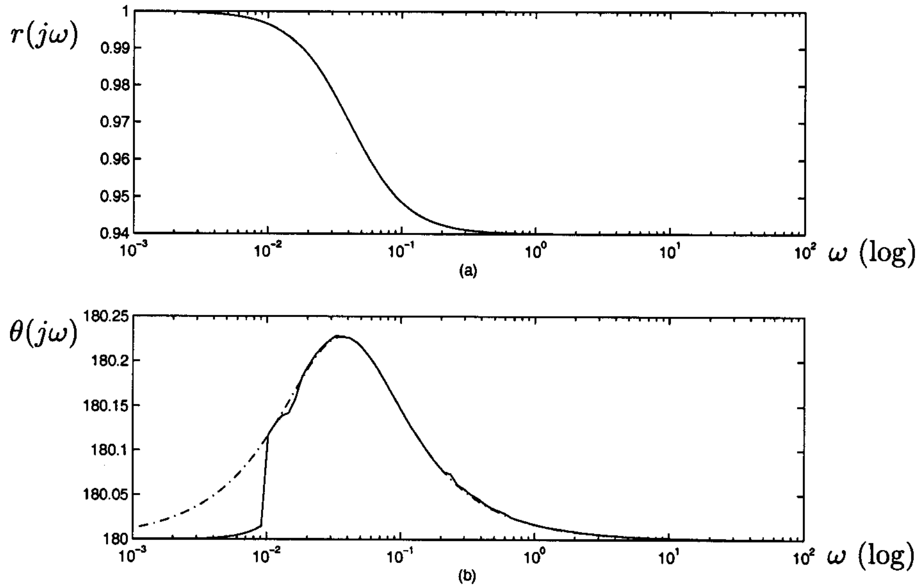

Figure 4. Values of $r(j\omega)$ and $\theta(j\omega)$ which maximizes $\tilde{\alpha}[r(j\omega),\theta(j\omega)]$ (—) and $\tilde{J}[r(j\omega),\theta(j\omega)]$ (–·–).

matrix (when the values of r and θ in $\gamma = r\,e^{j\theta}$ are such that K_P does not reduce the condition number of G) or its elements will be given by

$$k_{p_{11}} = 0, \qquad k_{p_{12}} = 1, \qquad k_{p_{21}} = \gamma(j\omega) \quad \text{and} \quad k_{p_{22}} = 0 \tag{69}$$

The values of $r(j\omega)$ and $\theta(j\omega)$ which minimize the condition number of the eigenvector matrix $\tilde{W}$ of $\tilde{G}$ and $\delta(\tilde{G})$ are shown in figure 4 (solid and dash–dotted lines, respectively). These values have been obtained in accordance with Algorithms 1 and 2 and correspond to the points of $\mathcal{R}$ for which $\tilde{\alpha}(r,\theta)$ and $\tilde{J}(r,\theta)$ attain their maximum. It is worth noting that both methods have produced approximately the same values for $r(j\omega)$ but, at the very low frequencies, $\theta(j\omega)$, obtained via maximization of $\tilde{\alpha}(r,\theta)$ differs slightly from those which maximize $\tilde{J}(r,\theta)$ (the curve for the latter is significantly smoother). For these values of $r(j\omega)$ and $\theta(j\omega)$, $\mathcal{C}(\tilde{W})$ gets very close to 1 and $\delta(\tilde{G})$ becomes approximately equal 0 (as shown in figures 2(b) (solid line) and 3(b)). This results show that the minimization problem together with the proposed precompensator have indeed turned a system with a badly skewed eigenvector matrix into one which is approximately normal for the whole frequency range, as was stated in Theorem 1. This is an important result since without using this precompensator the CLM could not be applied to the system. More importantly the designer now has a system with the best features for applying the CLM.

Once the frequency response for γ has been obtained, the next step is to find a stable and minimum phase transfer function for element $(2,1)$ of $K_P(s)$ whose frequency response matches that of $\gamma(j\omega)$ as closely as possible. However, a close look at figure 4 reveals that between $\omega = 10^{-3}$ and $\omega = 3 \times 10^{-2}$, $\theta(j\omega)$ increases from 180 to approximately 180.23 while $r(j\omega)$ decreases from 1 to 0.94. This implies that the polar plot of $\gamma(j\omega)$ will have an anticlockwise winding, and according to Horowitz and Ben-Adam (1989) there does not exist a stable transfer function whose frequency response matches that of $\gamma(j\omega)$. However, due to the near flatness of $r(j\omega)$ and $\theta(j\omega)$ it is enough to approximate $k_{p_{21}}(s)$ by a static transfer function $k_{p_{21}}(s) = -0.97$, which has been chosen since it corresponds to the arithmetic mean between the largest and the smallest values of $r(j\omega)$. The precompensator transfer function is therefore

$$K_P(s) = \begin{bmatrix} 0 & 1 \\ -0.97 & 0 \end{bmatrix} \tag{70}$$

The condition number of $\tilde{W}(j\omega)$ and the measure of normality $\delta[\tilde{G}(j\omega)]$ for the system precompensated with $K_P(s)$ given in (70) are depicted in figures 2(b) (dash–dotted line) and 3(c), respectively. They show that even for the precompensator whose frequency response of its $(2,1)$ element only approximates the one which actually minimizes $\mathcal{C}[W(j\omega)]$ and $\delta[\tilde{G}(j\omega)]$, the precompensation scheme proposed here has proved very effective.

4.2. *Example 2*

The transfer function of a linearized model of a vertical plane dynamics of an aircraft (Hung and MacFarlane 1982) is given by

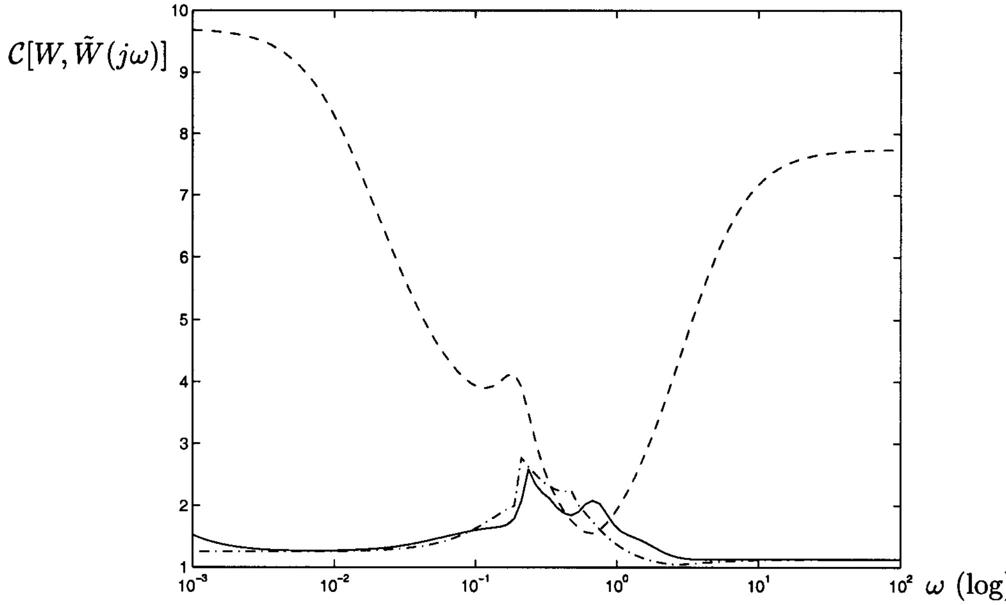

Figure 5. Condition number of $W(j\omega)$ (– –) and $\tilde{W}(j\omega)$ obtained after Step 5 of Algorithm 2 (– · –) and after frequency response fit of $K_P(s)$ (—).

$$G(s) = \frac{1}{d(s)} \begin{bmatrix} n_{11}(s) & n_{12}(s) & n_{13}(s) \\ n_{21}(s) & n_{22}(s) & n_{23}(s) \\ n_{31}(s) & n_{32}(s) & n_{33}(s) \end{bmatrix} \qquad (71)$$

where

$$d(s) = s^5 + 1.5953s^4 + 1.7572s^3 + 0.1112s^2 + 0.0561s$$

$$n_{11}(s) = -1.5750s^3 - 1.1190s^2 + 1.5409s - 0.0816$$

$$n_{12}(s) = 0.2909s^2 + 0.2527s + 0.3712$$

$$n_{13}(s) = 0.0732s^3 - 0.0646s^2 - 1.2125s - 0.0204$$

$$n_{21}(s) = -0.12s^4 - 0.0739s^3 - 0.5319s^2 - 0.2458s$$

$$n_{22}(s) = s^4 + 1.5415s^3 + 1.6537s^2$$

$$n_{23}(s) = -0.0052s^3 + 0.1570s^2 + 0.1828s$$

$$n_{31}(s) = 4.419s^3 + 1.6674s^2 + 0.1339s$$

$$n_{32}(s) = 0.0485s^2 + 0.3279s$$

$$n_{33}(s) = -1.6650s^3 - 1.1574s^2 - 0.0918s$$

In order for the CLM to be effective, it is first necessary to check if $G(s)$ is close to normal in the necessary frequency range. From figures 5 and 6 (dashed lines) it can be seen that, at low and high frequencies, $G(s)$ is far from normal, which shows the need for normalization. Note also that, $\delta[G(j\omega)] \to 0$ when $\omega \to \infty$. This calls our attention to the fact that at the very high frequency $G(j\omega)$ becomes normal. Note that this actually the case, since $G(j\omega) \to O$ when $\omega \to \infty$.

Since we are dealing now with a 3×3 system, Algorithm 1 cannot be applied. In accordance with Algorithm 2, there are $\binom{3}{2}$ pairs (k, l) that can be formed ($(1,2)$, $(1,3)$ and $(2,3)$), which implies that there are three possibilities of generating $K_{P_{kl}}(j\omega)$. The next step is to find, for each pair (k, l), the values of $r_{kl}(j\omega)$ and $\theta_{kl}(j\omega)$ which maximize $\tilde{J}_{kl}(j\omega)$. Figure 7(a–c) shows $\delta[G(j\omega)K_{P_{kl}}(j\omega)]$ for (k, l) equal $(1,2)$, $(1,3)$ and $(2,3)$, respectively, and also $\delta[G(j\omega)I_3]$ (figure 7(d)). The desired frequency response for $K_P(s)$ can then be obtained by choosing, for each frequency, the values of $k_{p_{ij}}(j\omega)$, $i, j = 1,2,3$ which make (among the four possibilities depicted in figure 7) $\tilde{G}(j\omega)$ closest to normal. Indeed, according to figure 7 the frequency response of $K_P(s)$ must be that of $K_{P_{12}}(j\omega)$ for frequencies below 0.21 rad/s and of $K_{P_{13}}(j\omega)$ for frequencies above 0.48 rad/s. In the narrow band from 0.21 to 0.48 rad/s a closer look of the values of $\delta[GK_{P_{kl}}(j\omega)]$ reveals that the frequency response of $K_P(s)$ must approximate that of $K_{P_{23}}(j\omega)$. The magnitudes and phase of $k_{p_{ij}}(j\omega)$, $i, j = 1,2,3$ for an ideal frequency response of $K_P(s)$ are shown, respectively, in figures 8 and 9 (+ +). Note that, for such a $K_P(j\omega)$ the eigenvector matrix condition number is smaller than 1.3 for most of the frequency range, being slightly greater than 2 for frequencies between 0.2 and 0.6 rad/s, as can be seen from figure 5 (dash–dotted line). This represents a significant improvement on the normality of $G(j\omega)$. Similar conclusions could be drawn from the analysis of $\delta[\tilde{G}(j\omega)]$ according to figure 6 (dash–dotted line).

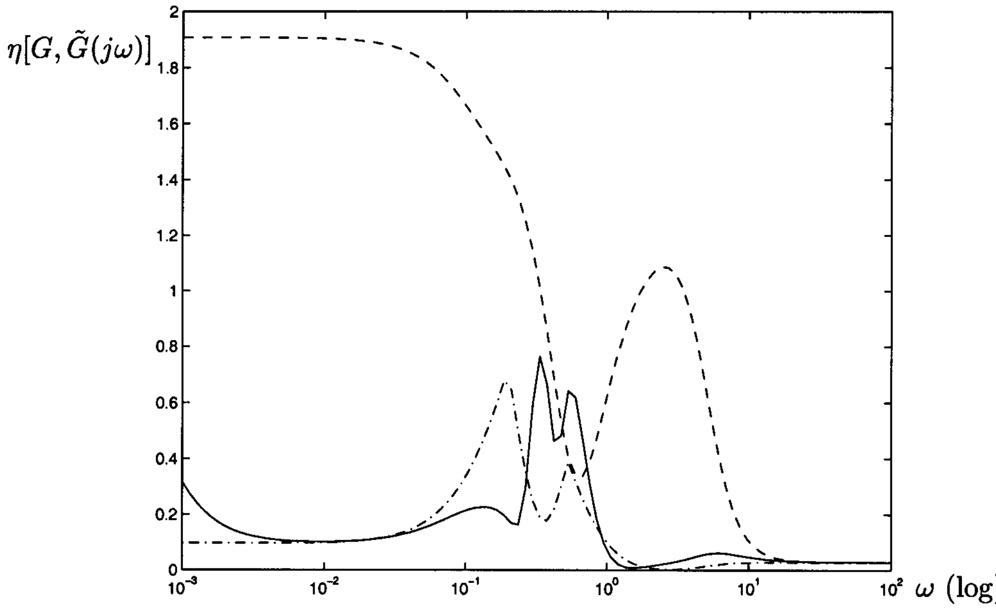

Figure 6. Measure of normality $\delta[G(j\omega)]$ (– –), $\delta[\tilde{G}(j\omega)]$ obtained after Step 5 of Algorithm 2 (– · –) and after frequency response fit of $K_P(s)$ (—).

The final step of algorithm 2 is to find stable transfer functions for the elements of $K_P(s)$ in such a way that their frequency responses closely match those obtained after Step 5. This can be done in a variety of ways, e.g. by designing Butterworth filters or simply by choosing appropriate poles and zeros. For example, a stable transfer function for element (1,2) can be obtained as follows: from figure 8, $k_{p_{12}}(j\omega)$ is either 1 for $\omega \le 0.187$ rad/s and 0 for $\omega \ge 0.187$ rad/s and from figure 9 the phase of $k_{p_{12}}(j\omega)$ is zero for the whole frequency range. These facts imply that the magnitude and phase requirements cannot be met simultaneously if one is sought a stable transfer function approximation for $k_{p_{12}}(j\omega)$. However, in trying to satisfy magnitude requirement, a closer look at figure 8 suggests the need for a pole at the proximity of 0.187 whose multiplicity will define the agreement between the magnitude of $k_{p_{12}}(j\omega)$ given in figure 8 and the magnitude of the frequency response of $k_{p_{12}}(s)$. For the sake of simplicity, a second order transfer function has been chosen, whose poles are –0.172 and –0.162, being therefore very close to the breakdown frequency $\omega = 0.187$ rad/s. The other transfer functions can be obtained in a similar way, leading to the following transfer matrix for $K_P(s)$

$$K_P(s) = \begin{bmatrix} \frac{0.1s}{s^2+0.1062s+0.0705} & \frac{0.0278}{s^2+0.3336s+0.0278} & \frac{s}{s+1.5129} \\ \frac{-0.6250s}{s^3+1.5807s^2+0.6259s+0.0010} & \frac{s}{s+3.8358} & \frac{0.3826s}{s^2+0.3806s+0.1789} \\ \frac{s}{s+1.0673} & \frac{0.2s}{s^2+0.2984s+0.0890} & \frac{0.0292}{s+0.0292} \end{bmatrix} \tag{72}$$

The frequency responses of each element of $K_P(s)$ are represented in figures 8 and 9 (solid lines). It is clear from figure 8 that there is a close agreement in the magnitudes but the target and the actual phases for the elements of $K_P(s)$ do not match quite well. The reasons for that are: (i) anticlockwise winding of the target frequency responses (elements (2,1) and (3,1)) and (ii) discontinuity of the target frequency responses. Despite these problems, it can been seen, from figures 5 and 6 (solid line), that the precompensator (72) has actually reduced drastically the condition number of the eigenvector matrix or equivalently has made the matrix $\tilde{G}(s)$ approximately normal. More importantly to say is that this has been achieved with a precompensator whose infinity norm is approximately 1—note from figure 10 that the largest singular values of $K_P(j\omega)$ varies from 0.87 to 1.42, being larger than 1.1 in a narrow frequency band (1.6×10^{-1} and 6×10^{-1} rad/s). It must be emphasized that precompensators with largest singular values closer to 1 in the frequency band above could probably be obtained at the expenses of an increase in the order of the transfer function of its elements.

4.3. *Comments*

In order to highlight the results presented in this section, the following comments are opportune:

(1) The precompensator given in (70) is very close to that given in (13), for which $r = 1$ and $\theta = \pi$. Therefore the reader could argue whether it is actually necessary to go through all the steps of Algorithms 1 and 2 if similar results could have

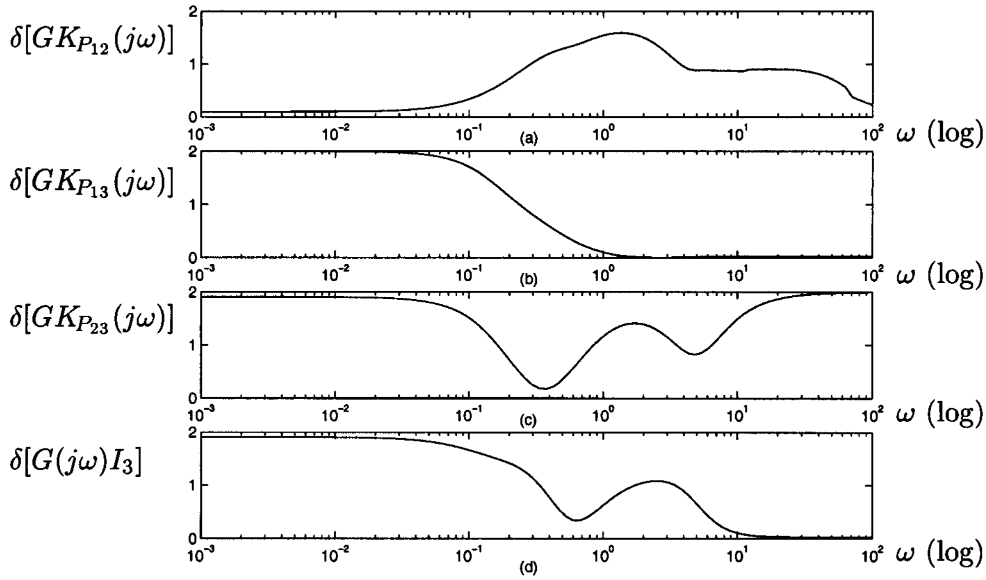

Figure 7. Measure of normality; (*a*) $\delta[GK_{P_{12}}(j\omega)]$; (*b*) $\delta[GK_{P_{13}}(j\omega)]$; (*c*) $\delta[GK_{P_{23}}(j\omega)]$; (d) $\delta[G(j\omega)]$.

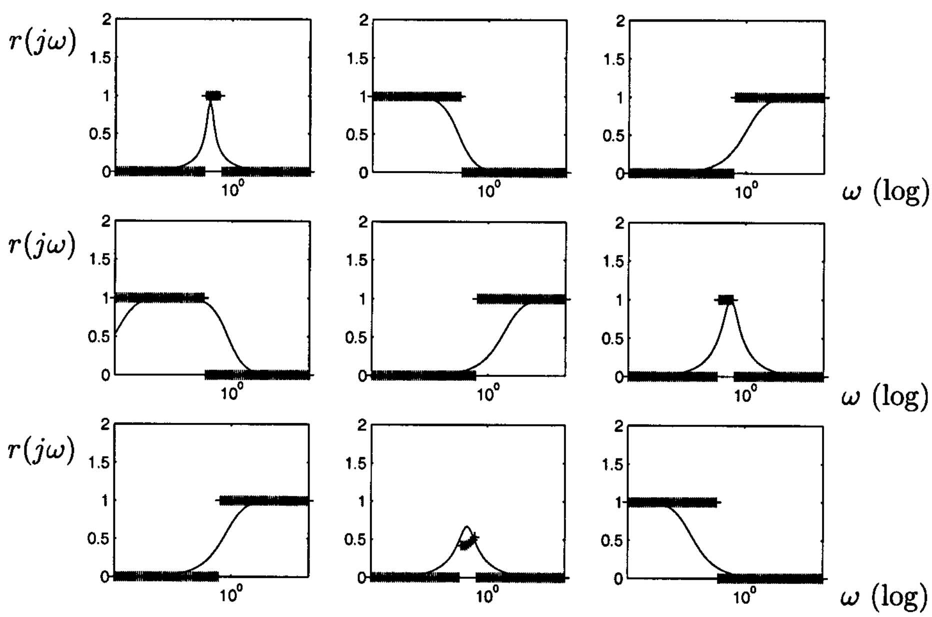

Figure 8. Magnitudes of $k_{p_{ij}}(j\omega)$, $i, j = 1, 2, 3$, for $K_P(s)$ obtained after Step 5 of Algorithm 2 (+ +) and after frequency response fit (—).

been obtained with a simpler precompensator. The answer to this question is given in Step 4 of Algorithm 1 and Step 5 of Algorithm 2, namely that, if for some frequency the plant is already close to normal then the precompensator frequency response should be close to an identity matrix at that frequency and therefore precompensator (13) would not be appropriate for this frequency. An example where this happens is the chemical reactor used in MacFarlane and Kouvaritakis (1977). The plant transfer matrix for the reactor is already close to normal for all the frequency range, but precompensator (13) was used to reduce interaction at high frequencies. Indeed the precompensator succeeded in reducing high frequency interaction but that was achieved at the expenses of an increase of the eigenvector matrix condition number at low frequencies (from 1.4 to approximately 10). It is important to note that this would be avoided if test (24) of Lemma 1 had been perfomed since $\rho(1, \pi) < 1$ for $\omega > 10$ and $\rho(1, \pi) > 1$ and $\omega < 10$ rad/s, being approximately 42.5 for the very low frequencies, showing the need for a dynamic

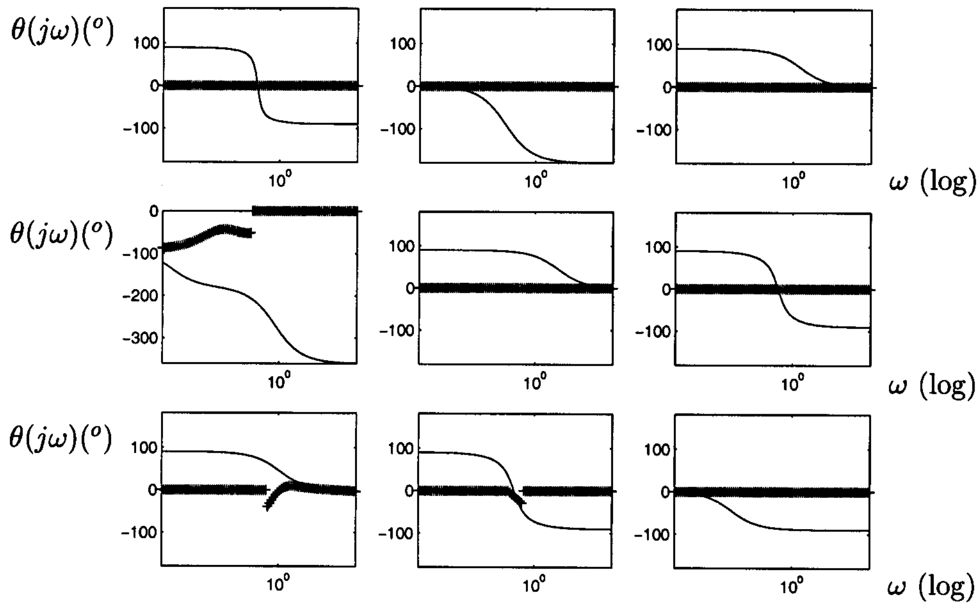

Figure 9. Phases of $k_{p_{ij}}(j\omega)$, $i, j = 1, 2, 3$, for $K_P(s)$ obtained after Step 5 of Algorithm 2 (+ +) and after frequency response fit (—).

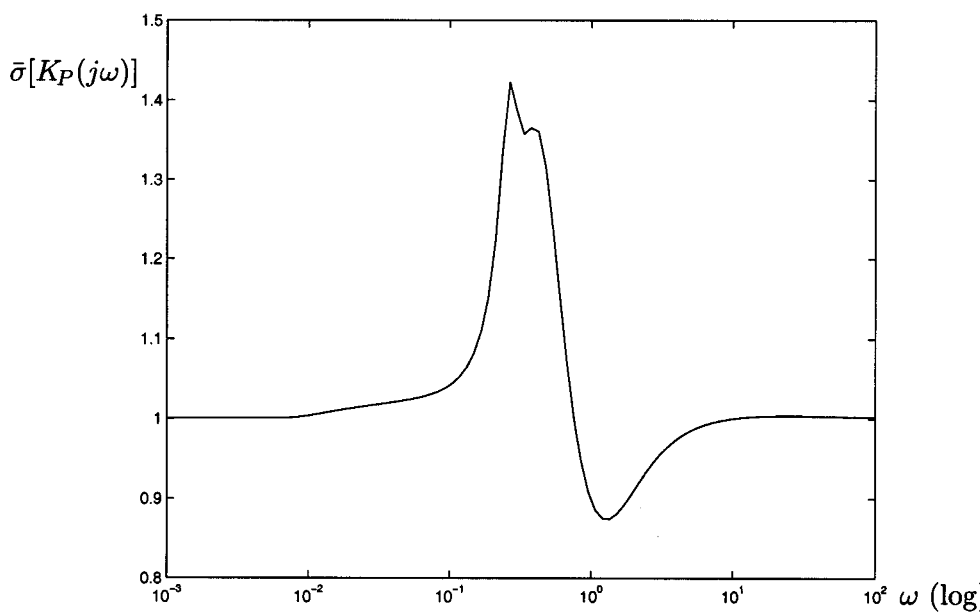

Figure 10. Maximum singular value of $K_P(j\omega)$ after response fit.

precompensator. This explains the problems of the use of the CLM on the design of a controller for the reactor plant.

(2) The impressive results obtained by precompensator (72) in Example 2 in spite of the poor agreement between the target and the actual phases of the frequency responses of the elements of $K_P(s)$ could lead the reader to conclude that the phase does not play an important role in the precompensator design proposed in this paper. That this is not necessarily true can be seen with the help of Example 1. In this case the phase of element $k_{p_{12}}(s)$ is always 180° since $k_{p_{12}}(s) = -0.97$, being therefore very close to the target frequency response phase which varies from approximately 180 to 180.23 (figure 4(*b*)). It is important to remark that if the frequency response of element $(2,1)$ had been chosen to be 0.97 (phase equal to 0), then the eigenvector condition number would be reduced to approximately 15 for most of the frequency range, which is much larger than that achieved by pre-

compensator (70). The importance of phase in a particular design can be viewed by plotting $\tilde{J}(\theta)$, at each frequency, for the value of r for which $\tilde{J}(r,\theta)$ achieves its maximum. The amount of variation of $\tilde{J}(\theta)$ provides information on the need for phase adjustment as well. In doing so for Example 2 above, the reader can see that $\tilde{J}(\theta)$ is indeed nearly flat, explaining the low importance of phase adjustment for that case.

5. Conclusion

In this paper the problem of precompensation of a multivariable plant with the view to making the precompensated system as normal as possible has been tackled. A precompensator structure has been proposed and the values of the frequency responses of its elements were calculated through the solution of two minimization problems: (i) the first one, suitable only for 2×2 systems aimed at reducing the condition number of the eigenvector matrix and (ii) the second one, which can be applied to any $m \times m$ system, had as cost a measure of normality. The proposed scheme has proved very efficient in both cases, as illustrated by two numerical examples taken from the literature.

Acknowledgment

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Appendix: proof of Theorem 2

Since $\tilde{J}_{kl}(r,\theta)$ given in (50) is continuously differentiable inside the region $\mathcal{R}$, it either attains its maximum at those pairs (r,θ) for which

$$\frac{\partial}{\partial r}\tilde{J}_{kl}(r,\theta) = 0 \quad \text{and} \quad \frac{\partial}{\partial \theta}\tilde{J}_{kl}(r,\theta) = 0 \tag{A1}$$

or occurs on the border of $\mathcal{R}$. The pairs on the border form the sets $\{(1,0)\}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ while the pairs inside $\mathcal{R}$ form the set $\mathcal{P}_3$.

Let us consider initially the points on the border of $\mathcal{R}$:

(i) For $\theta = 0$, $\tilde{J}_{kl}(r,\theta)$ becomes

$$\tilde{J}_{kl}(r) = \frac{b_0 r^4 + v_1 r^3 + b_2 r^2 + v_3 r + b_4}{a_0 r^4 + a_2 r^2 + a_4} \tag{A2}$$

Compute the derivative of $\tilde{J}(r)$ with respect to r, make it equal zero and since $r \in (0,1]$, we should select only those roots which are inside the interval, as stated in the definition of $\mathcal{P}_1$.

(ii) For the border $r = 1$, $\tilde{J}_{kl}(r,\theta)$ turns out to be

$$\tilde{J}_{kl}(\theta) = \frac{(v_1 + v_3)\cos\theta - (u_1 + u_3)\sin\theta + b_0 + b_2 + b_4}{a_0 + a_2 + a_4} \tag{A3}$$

Proceeding as in (i) leads to $\mathcal{P}_2$.

(iii) The last point on the border of $\mathcal{R}$ which can make $\tilde{J}(r,\theta)$ attain its maximum is $(1,0)$.

Let us now consider the interior points of $\mathcal{R}$. Computing $\partial\tilde{J}_{kl}(r,\theta)/\partial r = 0$, we obtain

$$\beta_0(\theta)r^6 + \beta_1 r^5 + \beta_2(\theta)r^4 + \beta_3 r^3 + \beta_4(\theta)r^2 + \beta_5 r + \beta_6(\theta) = 0 \tag{A4}$$

where

$$\left.\begin{aligned} \beta_0(\theta) &= a_0 b_1(\theta) \\ \beta_1 &= 2(a_0 b_2 - a_2 b_0) \\ \beta_2(\theta) &= 3a_0 b_3(\theta) - a_2 b_1(\theta) \\ \beta_3 &= 4(a_0 b_4 - a_4 b_0) \\ \beta_4(\theta) &= a_2 b_3(\theta) - 3a_4 b_1(\theta) \\ \beta_5 &= 2(a_2 b_4 - a_4 b_2) \\ \beta_6(\theta) &= -a_4 b_3(\theta) \end{aligned}\right\} \tag{A5}$$

Let us now consider $\partial\tilde{J}(r,\theta)/\partial\theta = 0$. It is straightforward to check that

$$r[b_1'(\theta)r^2 + b_3'(\theta)] = 0 \tag{A6}$$

where $b_3'(\theta)$ and $b_1'(\theta)$ denote, respectively, the derivatives of $b_3(\theta)$ and $b_1(\theta)$ with respect to θ.

(a) For $v_1 = u_1 = 0$ and either $v_3 \neq 0$ or $u_3 \neq 0$, then $b_3'(\theta) = -[v_3 \sin(\theta) + u_3 \cos(\theta)] = 0$ is the unique solution to equation (A6) in the interval $(0,1)$. Therefore if $v_3 = 0$ then the values of θ which satisfy (A6) will be $\pm\pi/2$. On the other hand, if $v_3 \neq 0$ then $\theta = \text{arc tg}(-u_3/v_3) + k\pi$, $k \in \mathbb{Z}$.

(b) When $v_3 = u_3 = 0$ and either $v_1 \neq 0$ or $u_1 \neq 0$, then $b_1'(\theta) = -[v_1 \sin(\theta) + u_1 \cos(\theta)] = 0$ is the unique solution to equation (A6) in the interval $(0,1)$. This implies that when $v_1 = 0$ then $\theta = \pm\pi/2$ and otherwise $\theta = \text{arc tg}(-u_1/v_1) + k\pi$, $k \in \mathbb{Z}$.

(c) If $v_1 = u_1 = v_3 = u_3 = 0$, then (A6) has an infinite number of solutions in the interval $(0,1)$.

(d) Under the assumption that either $v_1 \neq 0$ or $u_1 \neq 0$ and either $v_3 \neq 0$ or $u_3 \neq 0$, then $b_1'(\theta) \neq 0$ and $b_3'(\theta) \neq 0$ and therefore according to (A6) a value of r in the interval $(0,1)$ must satisfy

$$r^2 = -\frac{b_3'(\theta)}{b_1'(\theta)} = -\frac{u_3\cos\theta + v_3\sin\theta}{u_1\cos\theta + v_1\sin\theta} \tag{A7}$$

Defining $y = \sin\theta$ and $x = \cos\theta$, it is possible to write (A7) in terms of x and y as

$$r = \sqrt{-\frac{u_3x + v_3y}{u_1x + v_1y}} \tag{A8}$$

Substituting now (A8) in (A4) we obtain

$$\begin{aligned}&\phi_0x^8 + \phi_1x^7y + \phi_2x^6y^2 + \phi_3x^5y^3 + \phi_4x^4y^4 + \phi_5x^3y^5\\&\quad + \phi_6x^2y^6 + \phi_7xy^7 + \phi_8y^8 + \psi_0x^6 + \psi_1x^5y + \psi_2x^4y^2\\&\qquad + \psi_3x^3y^3 + \psi_4x^2y^4 + \psi_5xy^5 + \psi_6y^6 = 0\end{aligned} \tag{A9}$$

where ϕ_i, $i = 0,\dots,8$ are calculated as follows.

$$\phi_0 = (z_{14})^2 \tag{A10}$$

$$\begin{aligned}\phi_1 = {}& -2z_{14}(-3v_3u_3^2v_0 + u_0u_3^3 + u_3^2v_2v_1 - u_3^2u_1u_2\\& + 2u_3u_1v_2v_3 - 2u_3v_1u_1v_4 + u_3u_1^2u_4 - u_1^3u_6 - u_1^2v_3v_4\\& + 3v_1u_1^2v_6)\end{aligned} \tag{A11}$$

$$\begin{aligned}\phi_2 = {}& u_0^2u_3^6 - 12u_0v_0v_3u_3^5 + 15v_0^2v_3^2u_3^4 + (-20v_0v_2v_3^2u_3^3\\& - 2u_0u_2u_3^5 + 5z_1v_3u_3^4)u_1 + (z_1u_3^5 - 10v_0v_2v_3u_3^4)v_1\\& + (z_4u_3^4 - 4z_3v_3u_3^3 + 6z_2v_3^2u_3^2)u_1^2 - 2(z_3u_3^4 - 4z_2v_3u_3^3)\\& \times v_1u_1 + z_2u_3^4v_1^2 + (3z_7v_3u_3^2 - z_6u_3^3 - 3z_5v_3^2u_3)u_1^3\\& + 3(z_7u_3^3 - 3z_5v_3u_3^2)v_1u_1^2 - 3z_5u_3^3v_1^2u_1\\& + (z_{10}u_3^2 - 2z_{11}v_3u_3 + z_9v_3^2)u_1^4 + 6z_9u_3^2v_1^2u_1^2\\& - 4(z_{11}u_3^2 - 2z_9v_3u_3)v_1u_1^3 - 20v_4v_6u_3v_1^2u_1^3 - z_{12}u_1^5\\& + 5z_{13}v_1u_1^4 + u_6^2u_1^6 - 12u_6v_6v_1u_1^5 + 15v_6^2v_1^2u_1^4\end{aligned} \tag{A12}$$

$$\begin{aligned}\phi_3 = {}& 6u_0^2v_3u_3^5 + 20v_0^2v_3^3u_3^3 - 30u_0v_0v_3^2u_3^4 + (10z_1v_3^2u_3^3\\& - 20v_0v_2v_3^3u_3^2 - 10u_0u_2v_3u_3^4)u_1 + (4z_4v_3u_3^3 - 6z_3v_3^2u_3^2\\& + 4z_2v_3^3u_3)u_1^2 - (20v_0v_2v_3^2u_3^3 + 2u_0u_2u_3^5 - 5z_1v_3u_3^4)v_1\\& + 2(z_4u_3^4 - 4z_3v_3u_3^3 + 6z_2v_3^2u_3^2)v_1u_1\\& - (z_3u_3^4 - 4z_2v_3u_3^3)v_1^2 + 20v_6^2v_1^3u_1^3 - 3(-3z_7v_3u_3^2\\& + z_6u_3^3 + 3z_5v_3^2u_3)v_1u_1^2 + 3(z_7u_3^3 - 3z_5v_3u_3^2)v_1^2u_1\\& - 30u_6v_6v_1^2u_1^4 + (2z_{10}v_3u_3 - z_{11}v_3^2)u_1^4 + 4(z_{10}u_3^2\\& - 2z_{11}v_3u_3 + z_9v_3^2)v_1u_1^3 - z_5u_3^3v_1^3 - 4z_9u_3^2v_1^3u_1\\& - 5z_{12}v_1u_1^4 - 10z_{13}v_1^2u_1^3 - 20v_4v_6u_3v_1^3u_1^2 - 2u_4u_6v_3u_1^5\\& + 6u_6^2v_1u_1^5 + 6(z_{11}u_3^2 - 2z_9v_3u_3)v_1^2u_1^2 + (3z_7v_3^2u_3\\& - 3z_6v_3u_3^2 - z_5v_3^3)u_1^3\end{aligned} \tag{A13}$$

$$\begin{aligned}\phi_4 = {}& 15u_0^2v_3^2u_3^4 - 40u_0v_0v_3^3u_3^3 + 15v_0^2v_3^4u_3^2 + (-10v_0v_2v_3^4u_3\\& - 20u_0u_2v_3^2u_3^3 + 10z_1v_3^3u_3^2)u_1 + (10z_1v_3^2u_3^3\\& - 20v_0v_2v_3^3u_3^2 - 10u_0u_2v_3u_3^4)v_1 + (6z_4v_3^2u_3^2 - 4z_3v_3^3u_3\\& + z_2v_3^4)u_1^2 + 2(4z_4v_3u_3^3 - 6z_3v_3^2u_3^2 + 4z_2v_3^3u_3)v_1u_1\\& + (z_4u_3^4 - 4z_3v_3u_3^3 + 6z_2v_3^2u_3^2)v_1^2 + (z_7v_3^3 - 3z_6v_3^2u_3)u_1^3\\& - 3(3z_6v_3u_3^2 - 3z_7v_3^2u_3 + z_5v_3^3)v_1u_1^2 - 10z_{13}v_1^3u_1^2\\& - 10z_{12}v_1^2u_1^3 - 3(z_6u_3^3 - 3z_7v_3u_3^2 + 3z_5v_3^2u_3)v_1^2u_1\\& + (z_7u_3^3 - 3z_5v_3u_3^2)v_1^3 + 15u_6^2v_1^2u_1^4 + 15v_6^2v_1^4u_1^2\\& + z_{10}v_3^2u_1^4 + 4(2z_{10}v_3u_3 - z_{11}v_3^2)v_1u_1^3 + 6(z_{10}u_3^2\\& - 2z_{11}v_3u_3 + z_9v_3^2)v_1^2u_1^2 + z_9u_3^2v_1^4 - 4(z_{11}u_3^2\\& - 2z_9v_3u_3)v_1^3u_1 - 10v_4v_6u_3v_1^4u_1 - 10u_4u_6v_3v_1u_1^4\\& - 40u_6v_6v_1^3u_1^3\end{aligned} \tag{A14}$$

$$\begin{aligned}\phi_5 = {}& 20u_0^2v_3^3u_3^3 - 30u_0v_0v_3^4u_3^2 + 6v_0^2v_3^5u_3 + (5z_1v_3^4u_3\\& - 2v_0v_2v_3^5 - 20u_0u_2v_3^3u_3^2)u_1 - 5z_{13}v_1^4u_1 - (10v_0v_2v_3^4u_3\\& + 20u_0u_2v_3^2u_3^3 - 10z_1v_3^3u_3^2)v_1 + (4z_4v_3^3u_3 - z_3v_3^4)u_1^2\\& + 2(6z_4v_3^2u_3^2 - 4z_3v_3^3u_3 + z_2v_3^4)v_1u_1 + (4z_4v_3u_3^3\\& - 6z_3v_3^2u_3^2 + 4z_2v_3^3u_3)v_1^2 - 20u_4u_6v_3v_1^2u_1^3 - z_6v_3^3u_1^3\\& - 3(3z_6v_3^2u_3 - z_7v_3^3)v_1u_1^2 - (z_{11}u_3^2 - 2z_9v_3u_3)v_1^4\\& - 3(-3z_7v_3^2u_3 + 3z_6v_3u_3^2 + z_5v_3^3)v_1^2u_1 - (-3z_7v_3u_3^2\\& + z_6u_3^3 + 3z_5v_3^2u_3)v_1^3 + 4z_{10}v_3^2v_1u_1^3 + 6(2z_{10}v_3u_3\\& - z_{11}v_3^2)v_1^2u_1^2 - 2v_4v_6u_3v_1^5 - 10z_{12}v_1^3u_1^2 + (z_{10}u_3^2\\& - 42z_{11}v_3u_3 + z_9v_3^2)v_1^3u_1 - 30u_6v_6v_1^4u_1^2 + 20u_6^2v_1^3u_1^3\\& + 6v_6^2v_1^5u_1\end{aligned} \tag{A15}$$

$$\begin{aligned}\phi_6 = {}& 15u_0^2v_3^4u_3^2 - 12u_0v_0v_3^5u_3 + (-10u_0u_2v_3^4u_3 + z_1v_3^5)u_1\\& + 2(4z_4v_3^3u_3 - z_3v_3^4)v_1u_1 + (5z_1v_3^4u_3 - 2v_0v_2v_3^5\\& - 20u_0u_2v_3^3u_3^2)v_1 + z_4v_3^4u_1^2 + (6z_4v_3^2u_3^2 - 4z_3v_3^3u_3\\& + z_2v_3^4)v_1^2 - 3z_6v_3^3v_1u_1^2 - 3(3z_6v_3^2u_3 - z_7v_3^3)v_1^2u_1\\& - (-3z_7v_3^2u_3 + 3z_6v_3u_3^2 + z_5v_3^3)v_1^3 + v_0^2v_3^6\\& + 6z_{10}v_3^2v_1^2u_1^2 + 4(2z_{10}v_3u_3 - z_{11}v_3^2)v_1^3u_1 + (z_{10}u_3^2\\& - 2z_{11}v_3u_3 + z_9v_3^2)v_1^4 - 5z_{12}v_1^4u_1 - z_{13}v_1^5\\& - 20u_4u_6v_3v_1^3u_1^2 + 15u_6^2v_1^4u_1^2 - 12u_6v_6v_1^5u_1 + v_6^2v_1^6\end{aligned} \tag{A16}$$

$$\phi_7 = 2(-v_1 v_3^2 u_2 - v_1^3 u_6 + v_1^2 v_3 u_4 + v_3^3 u_0)(3v_3^2 u_3 u_0 - v_0 v_3^3 - v_3^2 u_2 u_1 + v_3^2 v_1 v_2 - v_3 v_1^2 v_4 - 2v_3 v_1 u_2 u_3 + 2v_3 u_4 v_1 u_1 - 3u_6 v_1^2 u_1 + v_1^2 u_3 u_4 + v_1^3 v_6) \quad (A17)$$

$$\phi_8 = (v_3^3 u_0 - v_3^2 u_2 v_1 + v_3 u_4 v_1^2 - u_6 v_1^3)^2 \quad (A18)$$

and ψ_j, $j = 0, \ldots, 6$ are given by

$$\psi_0 = u_3 u_1 (u_3^2 \beta_1 - u_3 \beta_3 u_1 + u_1^2 \beta_5)^2 \quad (A19)$$

$$\psi_1 = (u_3^2 \beta_1 - u_3 \beta_3 u_1 + u_1^2 \beta_5)(\beta_1 v_1 u_3^3 + 5\beta_1 v_3 u_1 u_3^2 - 3\beta_3 v_1 u_1 u_3^2 - 3u_3 \beta_3 v_3 u_1^2 + 5u_3 v_1 u_1^2 \beta_5 + v_3 u_1^3 \beta_5) \quad (A20)$$

$$\psi_2 = (3z_8 u_3^3 u_1 - 12\beta_3 \beta_5 u_3^2 u_1^2 + 10\beta_5^2 u_3 u_1^3 - 2\beta_1 \beta_3 u_3^4) v_1^2 + 10\beta_1^2 v_3^2 u_3^3 u_1 + 3z_8 v_3^2 u_3 u_1^3 + (-16\beta_1 \beta_3 v_3 u_3^3 u_1 + 5\beta_1^2 v_3 u_3^4 + 9z_8 v_3 u_3^2 u_1^2 + 5\beta_5^2 v_3 u_1^4 - 16\beta_3 \beta_5 v_3 u_3 u_1^3) v_1 - 12\beta_1 \beta_3 v_3^2 u_3^2 u_1^2 - 2\beta_3 \beta_5 v_3^2 u_1^4 \quad (A21)$$

$$\psi_3 = (-8\beta_3 \beta_5 u_3^2 u_1 + z_8 u_3^3 + 10\beta_5^2 u_3 u_1^2) v_1^3 + 10\beta_1^2 v_3^3 u_3^2 u_1 + z_8 v_3^3 u_1^3 + (-8\beta_1 \beta_3 v_3 u_3^3 + 9z_8 v_3 u_3^2 u_1 + 10\beta_5^2 v_3 u_1^3 - 24\beta_3 \beta_5 v_3 u_3 u_1^2) v_1^2 + (-24\beta_1 \beta_3 v_3^2 u_3^2 u_1 + 10\beta_1^2 v_3^2 u_3^3 + 9z_8 v_3^2 u_3 u_1^2 - 8\beta_3 \beta_5 v_3^2 u_1^3) v_1 - 8\beta_1 \beta_3 v_3^3 u_3 u_1^2 \quad (A22)$$

$$\psi_4 = (-2\beta_3 \beta_5 u_3^2 + 5\beta_5^2 u_3 u_1) v_1^4 + (3z_8 v_3 u_3^2 - 16\beta_3 \beta_5 v_3 u_3 u_1 + 10\beta_5^2 v_3 u_1^2) v_1^3 + 5\beta_1^2 v_3^4 u_3 u_1 + (-12\beta_1 \beta_3 v_3^2 u_3^2 + 9z_8 v_3^2 u_3 u_1 - 12\beta_3 \beta_5 v_3^2 u_1^2) v_1^2 + (10\beta_1^2 v_3^3 u_3^2 - 16\beta_1 \beta_3 v_3^3 u_3 u_1 + 3z_8 v_3^3 u_1^2) v_1 - 2\beta_1 \beta_3 v_3^4 u_1^2 \quad (A23)$$

$$\psi_5 = (\beta_1 v_3^3 u_1 + 5\beta_1 v_1 u_3 v_3^2 - 3v_3^2 \beta_3 v_1 u_1 + 5v_3 v_1^2 u_1 \beta_5 - 3v_3 \beta_3 u_3 v_1^2 + u_3 v_1^3 \beta_5)(v_3^2 \beta_1 - \beta_3 v_1 v_3 + \beta_5 v_1^2) \quad (A24)$$

$$\psi_6 = v_3 v_1 (v_3^2 \beta_1 - v_3 \beta_3 v_1 + v_1^2 \beta_5)^2 \quad (A25)$$

Note that the expressions for ϕ_i, $i = 0, \ldots, 8$ and ψ_j, $j = 0, \ldots, 6$ above depend on the variables z_k, $k = 1, \ldots, 14$ and u_l and v_l, $l = 0, 2, 4, 6$, which are given as

$$z_1 = 2(u_0 v_2 + v_0 u_2) \quad (A26)$$

$$z_2 = 2v_0 v_4 + v_2^2 \quad (A27)$$

$$z_3 = 2(u_2 v_2 + u_0 v_4 + v_0 u_4) \quad (A28)$$

$$z_4 = u_2^2 + 2u_0 u_4 \quad (A29)$$

$$z_5 = 2(v_0 v_6 + v_2 v_4) \quad (A30)$$

$$z_6 = 2(u_0 u_6 + u_2 u_4) \quad (A31)$$

$$z_7 = 2(u_0 v_6 + v_0 u_6 + u_2 u_4 + v_2 u_4) \quad (A32)$$

$$z_8 = \beta_3^2 + 2\beta_1 \beta_5 \quad (A33)$$

$$z_9 = 2v_2 v_6 + v_4^2 \quad (A34)$$

$$z_{10} = u_4^2 + 2u_2 u_6 \quad (A35)$$

$$z_{11} = 2(u_4 v_4 + u_2 v_6 + v_2 u_6) \quad (A36)$$

$$z_{12} = 2u_4 u_6 u_3 - 2(u_4 v_6 + v_4 u_6) v_3 \quad (A37)$$

$$z_{13} = 2v_4 v_6 v_3 - 2(u_4 v_6 + v_4 u_6) u_3 \quad (A38)$$

$$z_{14} = u_3^3 v_0 - u_3^2 v_2 u_1 - v_6 u_1^3 + u_3 v_4 u_1^2 \quad (A39)$$

$$v_0 = a_0 v_1, \quad v_2 = 3a_0 v_3 - a_2 v_1, \quad v_4 = a_2 v_3 - 3a_4 v_1 \quad \text{and} \quad v_6 = -a_4 v_3 \quad (A40)$$

$$u_0 = a_0 u_1, \quad u_2 = 3a_0 u_3 - a_2 u_1, \quad u_4 = a_2 u_3 - 3a_4 u_1 \quad \text{and} \quad u_6 = -a_4 u_3 \quad (A41)$$

with u_1, v_1, u_3 and v_3 being given by equations (59), (60), (62) and (63).

The definition of x and y implies that $x^2 = 1 - y^2$. Thus, substituting x in $x^2 = 1 - y^2$ in (A9) leads to the equation

$$\tau_0 y^{16} + \tau_1 y^{14} + \tau_2 y^{12} + \tau_3 y^{10} + \tau_4 y^8 + \tau_5 y^6 + \tau_6 y^4 + \tau_7 y^2 + \tau_8 = 0 \quad (A42)$$

where

$$\tau_0 = (\phi_0 - \phi_2 + \phi_4 - \phi_6 + \phi_8)^2 + (\phi_3 - \phi_1 - \phi_5 + \phi_7)^2 \quad (A43)$$

$$\tau_1 = (\phi_3 - \phi_1 - \phi_5 + \phi_7)(7\phi_1 - 5\phi_3 + 3\phi_5 - \phi_7 + 2\psi_1 - 2\psi_3 + 2\psi_5) + 2(\psi_2 + \phi_6 - 2\phi_4 - 4\phi_0 - \psi_0 + \psi_6 + 3\phi_2 - \psi_4)(\phi_0 - \phi_2 + \phi_4 - \phi_6 + \phi_8) \quad (A44)$$

$$\tau_2 = 2(\psi_2 + \phi_6 - 2\phi_4 - 4\phi_0 - \psi_0 + \psi_6 + 3\phi_2 - \psi_4)^2 + 2(-3\phi_1 - 2\psi_1 + \phi_3 + \psi_3)(\phi_3 - \phi_1 - \phi_5 + \phi_7) + (3\phi_1 + \phi_5 + \psi_1 - 2\phi_3 - \psi_3 + \psi_5)(5\phi_1 - 4\phi_3 + 3\phi_5 - 2\phi_7 + \psi_1 - \psi_3 + \psi_5) + 2(3\psi_0 - 2\psi_2 + \phi_4 + \psi_4 + 6\phi_0 - 3\phi_2)(\phi_0 - \phi_2 + \phi_4 - \phi_6 + \phi_8) \quad (A45)$$

$$\tau_3 = 2(3\psi_0 - 2\psi_2 + \phi_4 + \psi_4 + 6\phi_0 - 3\phi_2)(\psi_2 + \phi_6 - 2\phi_4 - 4\phi_0 - \psi_0 + \psi_6 + 3\phi_2 - \psi_4) + 2(\phi_1 + \psi_1)$$

$$\times (\phi_3 - \phi_1 - \phi_5 + \phi_7) + 2(-4\phi_0 - 3\psi_0 + \phi_2 + \psi_2) \times (\phi_0 - \phi_2 + \phi_4 - \phi_6 + \phi_8) - 2(3\phi_1 + 2\psi_1 - \phi_3 - \psi_3)(4\phi_1 - 3\phi_3 + 2\phi_5 - \phi_7 + \psi_1\psi_3 + \psi_5) - (3\phi_1 + \phi_5 + \psi_1 - 2\phi_3 - \psi_3 + \psi_5)^2 \quad \text{(A46)}$$

$$\tau_4 = (3\psi_0 - 2\psi_2 + \phi_4 + \psi_4 + 6\phi_0 - 3\phi_2)^2 + (-3\phi_1 - 2\psi_1 + \phi_3 + \psi_3)^2 + 2(\phi_1 + \psi_1) \times (4\phi_1 - 3\phi_3 + 2\phi_5 - \phi_7 + \psi_1 - \psi_3 + \psi_5) - 2(-3\phi_1 - 2\psi_1 + \phi_3 + \psi_3)(3\phi_1 + \phi_5 + \psi_1 - 2\phi_3 - \psi_3 + \psi_5) + 2(\phi_0 + \psi_0)(\phi_0 - \phi_2 + \phi_4 - \phi_6 + \phi_8) + 2(-4\phi_0 - 3\psi_0 + \phi_2 + \psi_2)(\psi_2 + \phi_6 - 2\phi_4 - 4\phi_0 - \psi_0 + \psi_6 + 3\phi_2 - \psi_4) \quad \text{(A47)}$$

$$\tau_5 = 2(\phi_0 + \psi_0)(\psi_2 + \phi_6 - 2\phi_4 - 4\phi_0 - \psi_0 + \psi_6 + 3\phi_2 - \psi_4) + 2(-4\phi_0 - 3\psi_0 + \phi_2 + \psi_2)(3\psi_0 - 2\psi_2 + \phi_4 + \psi_4 + 6\phi_0 - 3\phi_2) - 2(\phi_1 + \psi_1)(6\phi_1 - 3\phi_3 + \phi_5 + 3\psi_1 - 2\psi_3 + \psi_5) - (-3\phi_1 - 2\psi_1 + \phi_3 + \psi_3)^2 \quad \text{(A48)}$$

$$\tau_6 = 2(\phi_0 + \psi_0)(3\psi_0 - 2\psi_2 + \phi_4 + \psi_4 + 6\phi_0 - 3\phi_2) + (-4\phi_0 - 3\psi_0 + \phi_2 + \psi_2)^2 + (\phi_1 + \psi_1) \times (7\phi_1 - 2\phi_3 + 5\psi_1 - 2\psi_3) \quad \text{(A49)}$$

$$\tau_7 = 2(\phi_0 + \psi_0)(-4\phi_0 - 3\psi_0 + \phi_2 + \psi_2) - (\phi_1 + \psi_1)^2 \quad \text{(A50)}$$

$$\tau_8 = (\phi_0 + \psi_0)^2 \quad \text{(A51)}$$

It is worth noting that equation (A42) has only even powers of y. This allows us to reduce the equation degree providing we carry out the substitution $t = y^2$, leading to the defining equation of set $\mathcal{P}_3$ (third possibility). Finally notice that the real roots of (A42) between 0 and 1 are the only ones that can actually be points of the domain for which $\tilde{J}(r, \theta)$ may attain its maximum since $y = \sin\theta = \pm\sqrt{t}$ and $x = \cos\theta = \pm\sqrt{1-t}$. This completes the proof of Theorem 2. □

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