

## Design of causal reversed-frame-normalizing controllers using bicausal expansions

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Reverse-frame-normalizing controllers overcome the sensitivity problems of commutative controllers and balance the tolerance of multivariable feedback systems to input and output multiplicative unstructured uncertainty. They are, however, based on the singular value decomposition of transfer function matrices and lead to implementation difficulties on account of the irrational nature of the decomposition. Realizable approximations can be derived through the use of least-squares frequency response fit algorithms, but these are either nonlinear or require an *a priori* definition of the controller poles. In this paper we deploy a bicausal sequence representation for the singular value decomposition and derive conditions that overcome anticausality difficulties. This treatment leads to a characterization of the whole class of controllers and proves that frequency response targets for the generalized Nyquist diagrams cannot be defined arbitrarily. Finally we propose an algorithm for the systematic trade-off between the objective of achieving normality and that of reaching specific frequency response targets.

### 1. Introduction

The generalized Nyquist criterion (MacFarlane and Postlethwaite 1977) gives necessary and sufficient conditions for the stability of linear multivariable systems and provides the means for the analysis of multivariable closed-loop behaviour. A corresponding approach to the design of multivariable controllers  $K(z)$  is made possible through commutativity (MacFarlane and Belletrutti 1973); our interest in this paper is in discrete time systems only—hence the use of  $z$  (rather than Laplace) transforms. However, commutative controllers are based on eigenvector functions which are irrational and therefore defy implementation. To overcome this difficulty it is possible to use either constant approximations (MacFarlane and Kouvaritakis 1977), or causal power series approximations (Cloud and Kouvaritakis 1987). The former leads to simple controllers but these are only approximately commutative and are only useful over a limited range of frequencies, whereas the latter works well for the case when eigenfunctions of the plant transfer function matrix  $G(z)$  have stable branch points only. Recent work (Kouvaritakis and Basilio 1994) introduced the idea of an eigenvector approximation by bicausal power series and thus proposed an effective approach to the design of commutative controllers for the general case. These bicausal series approximations are based on truncated Laurent series expansions of eigenvector functions which are assumed to be analytic on an annulus including the unit circle centred at the origin of the  $z$  plane. The advantage here is that, by increasing the length of the truncated series, one can obtain approximations to the frequency response of the eigenvector functions which are as accurate as desired.

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Eigenvalue sensitivity problems limit the practical value of commutative controllers, especially in the case of skew eigenvectors. This problem does not arise if the eigenvectors of  $G(z)K(z)$  are orthogonal, namely if  $G(z)K(z)$  is normal. Furthermore a controller which achieved normality for  $G(z)K(z)$  and  $K(z)G(z)$  simultaneously would have the effect of balancing the tolerance of the feedback system to both input and output multiplicative unstructured uncertainty in  $G(z)$ . These considerations form the motivation behind the reverse-frame-normalizing controller (RFNC) design (Hung and MacFarlane 1982).

RFNCs are based on singular values of vectors which, like eigenfunctions, lead to implementation difficulties on account of their irrational nature. A possible way around this is to derive the frequency response of an ideal (exact) RFNC and approximate this in a suitable least-squares sense. The most general approach to this approximation leads to a nonlinear algorithm which may run up against convergence problems and may result in unstable controllers which do not satisfy the generalized Nyquist criterion. These difficulties may be overcome if one is prepared to define *a priori* the controller poles, but this amounts to giving up design freedom which may be needed for the attainment of the RFNC objectives: firstly normality and secondly reaching pre-specified frequency response targets for the generalized Nyquist diagrams of  $G(z)K(z)$ .

In this paper we introduce bicausal sequence representations for the singular vector frames and derive conditions on the controller structure that overcomes anticausality difficulties. On the basis of these we are able to design causal controllers  $K(z)$  which achieve reverse frame alignment and hence ensure the normality of  $G(z)K(z)$  and  $K(z)G(z)$ . This analysis exposes some features of RFNC that were not clear before, namely that frequency response targets for the generalized Nyquist diagrams of  $G(z)K(z)$  cannot be defined arbitrarily; not all targets are achievable by an exact or even approximate RFNC. We propose an algorithm which allows the designer to reach a systematic compromise between the objective of achieving normality and that of reaching particular frequency response targets. The superiority of the proposed algorithm over the earlier algorithms (Hung and MacFarlane, 1982) is illustrated by means of a numerical example.

## 2. Background

### 2.1. The quasi-Nyquist decomposition

We begin our exposition by introducing the quasi-Nyquist decomposition and to start with we discuss the motivation behind; for a somewhat different interpretation the reader is referred to Hung and MacFarlane (1982). Let  $G \in G^{m \times m}$  denote the value of  $G(z)$  for some  $z = \exp\{i\omega T\}$ , where  $T$  denotes the sampling interval and  $\omega$  is some frequency and consider the eigenvalue and singular value decompositions of  $G$  given as

$$G = W\Lambda V = W\Lambda(V^{-1})^{-1} \quad (1a)$$

$$(V^{-1})^{-1}W = I_m \quad (1b)$$

$$G = X\Sigma Y^* \quad (2a)$$

$$X^*X = I_m \quad (2b)$$

$$Y^*Y = I_m \quad (2c)$$

where  $(.)^*$  denotes transposition and complex conjugation and in the case of unitary

matrices, such as  $X$  and  $Y$ , is equivalent to inversion. Both decompositions have the same structure in that  $G$  is expressed as the product of a matrix of column vectors, a diagonal matrix and the inverse of a matrix of column vectors; for convenience the matrices of vectors will be referred to as frames and in particular  $W$  will be called the eigenvector frame, whereas  $X$  and  $Y$  will be called the output and input singular vector frames. There are two important differences between these decompositions: firstly, the input and output singular vectors form an orthonormal set of vectors, whereas the eigenvectors do not; secondly  $V^{-1}$  is aligned with  $W$  (see (1 b)) whereas  $X$  and  $Y$  are not since in general  $Y^{-1}X = Y^*X \neq I_m$ . On account of the alignment property the eigenvalues of  $G$  contain information which determines closed-loop stability (through the generalized Nyquist criterion), and on account of the orthogonality of  $X$  and  $Y$  the singular value does not suffer the sensitivity problems of eigenvalues. To put it another way,  $\Sigma$  is not sensitive to perturbations on  $G$  but could be a long way away from  $\Lambda$  and thus does not convey useful information about stability.

The ideal decomposition in this context then must retain the orthogonality property but must at the same time optimize as far as possible the alignment property:

$$G = P\Gamma Q^* \quad (3a)$$

$$P^*P = I_m \quad (3b)$$

$$Q^*Q = I_m \quad (3c)$$

$$Q^{-1}P = Q^*P \approx I_m \quad (3d)$$

Equations (3a), (3b) and (3c) combine with (2) to give

$$\Gamma\Gamma^* = [P^*X]\Sigma^2[P^*X]^{-1} \quad (4a)$$

$$\Gamma^*\Gamma = [(Y^*Q)^*]\Sigma^2[(Y^*Q)^*]^{-1} \quad (4b)$$

which therefore implies that

$$P = X \exp\{-j\Phi\}, \quad Q = Y \exp\{-i\Psi\}, \quad \Phi = \text{diag}\{\phi_i\}, \quad \Psi = \text{diag}\{\psi_i\}, \quad i = 1, 2, \dots, m \quad (5)$$

Introducing these into (3) we get

$$G = X \exp\{-j\Theta\} \Gamma Y^* \quad (6a)$$

$$Y^*X \exp\{-j\Theta\} \approx I_m \quad (6b)$$

where  $\Theta = \Phi - \Psi$ .

Clearly  $\Theta$  must be chosen so as to minimize the error in the alignment condition (6b) and an obvious way to choose it is to minimize the measure of 'misalignment' defined below:

$$J_{X,Y} = \min_{\Theta = \text{diag}\{\theta_i\}} \|Y^*X \exp\{-j\Theta\} - I_m\|_2 \quad (7)$$

where  $\|\cdot\|_2$  denotes the spectral norm of a matrix. Rewriting (6) for the minimizing  $\Theta$ , say  $\Theta^0$  we derive the quasi-Nyquist decomposition:

$$G = U\Gamma Y^* \quad (8a)$$

$$U = X \exp\{-j\Theta^0\} \quad (8b)$$

$$\Gamma = \exp\{j\Theta^0\} \Sigma \quad (8c)$$

It is interesting to note that, unlike  $\Sigma$ ,  $\Gamma$  contains phase information and therefore is suitable for use in a Nyquist type of approach; this together with the arguments given above justify the term quasi-Nyquist.

## 2.2. The reversed-frame-normalizing controller

The decomposition of (8) is written for one frequency only but can be written for a general  $z$  on the unit circle as

$$G(z) = U(z)\Gamma(z)Y^*(z), \quad \Gamma(z) = \text{diag}\{\gamma_i(z)\}, \quad i = 1, 2, \dots, m \quad (9)$$

Assume next that the input vector into  $G(z)$  is the output vector of a controller  $K(z)$  such that the compensated transfer function matrix is  $Q(z) = G(z)K(z)$ , and let unity feedback be applied around  $Q(z)$ . To balance the tolerance of such a feedback system to unstructured multiplicative input and output uncertainty on  $G(z)$ , RFNCs seek to equalize the singular values of  $[I_m + K(z)G(z)]^{-1}$  and  $[I_m + G(z)K(z)]^{-1}$  which can be achieved if and only if  $G(z)K(z)$  and  $K(z)G(z)$  are both normal; this in turn can be achieved if and only if

$$K(z) = Y(z)\Gamma_K(z)U^*(z) \quad (10a)$$

$$\Gamma_K(z) = \text{diag}\{k_i(z)\}, \quad i = 1, 2, \dots, m \quad (10b)$$

where  $k_i(z)$  are scalar transfer functions to be selected by the designer. The decomposition of (10a) is in fact a quasi-Nyquist decomposition because both  $Y$  and  $U$  are unitary on the unit circle and at each frequency it is assumed that  $U = X \exp\{-j\Theta^0\}$  where  $\Theta^0$  is chosen so as to give the best possible alignment between  $Y$  and  $U$ . Clearly  $K(z)$  results in a normal  $Q(z) = U(z)\Gamma(z)\Gamma_K(z)U^*(z)$ ; this together with the fact that the order in which  $Y$  and  $U$  appear in  $K(z)$  is the reverse of that of the quasi-Nyquist decomposition of  $G(z)$  justify the term ‘reversed-frame-normalizing controller’.

$Y(z)$  can be shown to be the eigenvector matrix of  $G^*(z)G(z)$  and hence in general will be an irrational function of  $z$ ; furthermore, for  $m > 2$ ,  $Y(z)$  will not be known explicitly. Similarly  $U(z)$  depends on  $X(z)$  which can be shown to be the eigenvector matrix of  $G(z)G^*(z)$  and hence is subject to the same difficulties as  $Y(z)$ ; in addition to this,  $\Theta^0(z)$  depends on  $X(z)$  and  $Y(z)$  which are irrational and, for  $m > 2$ ,  $\Theta^0(z)$  will not be known explicitly and can only be computed numerically at each frequency. Thus the expression for  $K(z)$  in (10a) defies implementation. As a remedy to this Hung and MacFarlane suggested that the ideal RFNC of (10a) be computed at a suitably large number of frequency points and a rational approximation  $K_a(z)$  be found to give an appropriate frequency response fit. In particular let the  $K_a(z) = D^{-1}(z)N(z)$  where  $D(z)$  and  $N(z)$  are polynomial matrices and  $D(z)$  is given; then the coefficients of  $N(z)$  can be computed so as to minimize a suitable measure of the error between the ideal compensated  $Q_0(z) = U(z)\Gamma(z)\Gamma_K(z)Y^*(z)$  and  $G(z)K_a(z)$  evaluated at a set of pre-selected frequency points and added over such points:

$$J = \sum_{i=1}^n \|G(z_i)D^{-1}(z_i)N(z_i) - Q_0(z_i)\|_F^2 \quad (11)$$

where the subscript F denotes the Frobenius norm (with possible weighting) and  $z_i = \exp\{j\omega_i T\}$  with  $\omega_i$  denoting the pre-selected frequencies. The above cost is quadratic in each of the elements of the matrix coefficients of  $N(z)$  and hence minimization can be performed explicitly by solving stationarity conditions (which are linear).

The above algorithm presupposed knowledge of  $D(z)$ . In general it is not obvious how to best choose  $D(z)$ , in which case it is necessary to include the coefficients of  $D(z)$

into the minimization problem which now becomes nonlinear and therefore much harder to solve and convergence to the global optimum may no longer be guaranteed. More importantly one loses control over the position of the poles of  $K_a(z)$ .

### 3. Causal reversed-frame-normalizing controllers

#### 3.1. Bicausal sequence representation of the quasi-Nyquist decomposition

Scaling  $G(z)$  by  $d(z)$ , the least-common-denominator polynomial defines the numerator polynomial matrix  $N(z) = d(z)G(z)$  but does not affect the vector frames involved in the singular value and quasi-Nyquist decompositions:

$$N(z) = X(z)\Sigma_N(z)Y^*(z) = U(z)\Gamma_N(z)Y^*(z), \quad \Sigma_N(z) = \text{diag}\{n_i(z)\}, \quad \Gamma_N(z) = d(z)\Gamma(z) \quad (12)$$

where  $n_i(z)$  denote the singular value functions of  $N(z)$  given by the square root of the branches of the characteristic function, say  $\gamma(z)$ , of  $N^*(z)N(z)$ . It is noted that  $N(z)$  does not have any poles and thus is more convenient for both analysis and the computation of the frames  $X$ ,  $Y$  and  $U$ .

**Lemma 3.1:** *Let  $\mathbf{x}_i(z)$  and  $\mathbf{y}_i(z)$  denote respectively the input and output principal directions of  $N(z)$  associated with the  $i$ th singular value  $n_i(z)$ . Then  $n_i(z)$ ,  $\mathbf{x}_i(z)$  and  $\mathbf{y}_i(z)$  admit bicausal expansions in terms of positive and negative powers of  $z$  of the form:*

$$n_i(z) = \sum_{k=0}^{\infty} n_{i_k}(z^k + z^{-k}), \quad \mathbf{x}_i(z) = \sum_{k=-\infty}^{\infty} \mathbf{z}_{i_k} z^{-k}, \quad \mathbf{y}_i(z) = \sum_{k=-\infty}^{\infty} \mathbf{y}_{i_k} z^{-k} \quad (13)$$

which represents  $n_i(z)$ ,  $\mathbf{x}_i(z)$  and  $\mathbf{y}_i(z)$  on the unit circle. Moreover the sequences of  $\{2n_{i_0} n_{i_1} n_{i_2}, \dots\}$ ,  $\{\mathbf{x}_{i_0}, \mathbf{x}_{i_1}, \dots\}$ ,  $\{\mathbf{x}_{i_{-1}}, \mathbf{z}_{i_{-2}}, \dots\}$ ,  $\{\mathbf{y}_{i_0}, \mathbf{y}_{i_{-1}}, \dots\}$ ,  $\{\mathbf{y}_{i_{-1}}, \mathbf{y}_{i_{-2}}, \dots\}$  all converge to zero.

**Proof:** This is as per Kouvaritakis *et al.* (1993, Theorem 3.2) except that here we have removed the assumption that the characteristic function  $\tau(z)$  of  $N^*(z)N(z)$  should have no branch points on the unit circle and should not have any branch cuts that cross the unit circle. The justification for the latter is provided by Corollary 3 of Kouvaritakis and Rossiter (1991) according to which no branch cut of  $\tau(z)$  can cross the unit circle since the eigenvalues of  $N^*(1)N(1)$  and  $N^*(-1)N(-1)$  are all real. Branch points, on the other hand, are associated with non-simple Jordan forms (Cloud and Kouvaritakis 1987) and thus those of  $\tau(z)$  cannot exist on the unit circle because  $N^*(z)N(z)$  is hermitian for  $z = \exp\{j\omega T\}$ , and hence has simple Jordan form.  $\square$

**Lemma 3.2:** The  $i$ th column vector of the factor  $U(z)$  in the quasi-Nyquist decomposition of  $N(z)$  (and hence  $G(z)$ ) admits a bicausal expansion in terms of positive and negative powers of  $z$  of the form

$$\mathbf{u}_i(z) = \sum_{k=-\infty}^{\infty} \mathbf{u}_{i_k} z^{-k} \quad (14)$$

which represents  $\mathbf{u}_i(z)$  on the unit circle. Moreover the sequences of the coefficients of the causal and anticausal parts,  $\{\mathbf{u}_{i_0}, \mathbf{u}_{i_1}, \dots\}$  and  $\{\mathbf{u}_{i_{-1}}, \mathbf{u}_{i_{-2}}, \dots\}$ , converge to zero.

**Proof:** From (8b) we have that  $\mathbf{u}(z) = \mathbf{x}(z) \exp\{-j\theta_i^0(z)\}$  and by Lemma 3.1 we know that  $\mathbf{x}_i(z)$  admits a bicausal expansion; so it remains to show the same for  $f_i(z) =$

$\exp\{-j\theta_i^0(z)\}$ . By Laurent's theorem,  $f_i(z)$  will have a bicausal representations whose causal and anticausal sequences converge to zero if it is analytic inside an annulus containing the unit circle. However, for  $z = \exp\{-j\omega T\}$ ,  $f_i(z)$  lies on the unit circles for all  $\omega T \in [0, 2\pi)$ ; hence to prove analyticity all we need to do is to establish continuity, but this is implied by the definition of  $\theta^0$  and the continuity of  $X$ .  $\square$

The computation of the causal and anticausal sequences of  $Y(z)$  and  $U(z)$  can be performed by a process of frequency sampling and inverse discrete Fourier transformations as stated below.

**Algorithm 3.1:**

- Step 1.* Let  $\mu+1$  and  $\mu$  denote the number of significant elements in the sequence for  $Y(z)$  and  $U(z)$  and assign a suitably large value to  $\mu$ .
- Step 2.* Let  $h(z)$  denote a particular element of either  $Y(z)$  or  $U(z)$  and evaluate  $h(z)$  at points  $z_k = \exp\{-j2\pi k/(2\mu+1)\}$  for  $k = 0, 2, \dots, 2\mu$  and enter the results in sequence as elements of a vector  $\hat{\mathbf{h}}$ . Then obtain the inverse discrete Fourier transform of  $h(z)$  by computing the vector

$$\mathbf{h} = \frac{1}{2\mu+1} F^* \hat{\mathbf{h}} \quad (15)$$

$F$  is a  $(2\mu+1) \times (2\mu+1)$  matrix whose  $p+1, q+1$  element is  $\exp\{j2\pi pq/(2\mu+1)\}$ .

- Step 3.* Derive the anticausal sequence of  $h(z)$  by reversing the order of the last  $\mu$  elements of  $\mathbf{h}$ ; the causal sequence is given by the first  $\mu+1$  elements (taken in the order they appear).

**Remark 3.1:** In theory  $\mu$  should be taken to be arbitrarily large, but in practice, owing to the convergence to zero of both causal and anticausal sequences, this is not necessary. As a consequence, however, this truncation will incur an error which of course can be made as small as desired by increasing  $\mu$ . Henceforth all results stated for finite sequence will be true to within these truncation errors; this fact will be assumed and therefore will not be repeated in the sequel.  $\square$

The theoretical background and justification for this algorithm are similar to those given by Kouvaritakis and Rossiter (1991) for the computation of the bicausal sequence representation of eigenvalue decomposition of transfer function matrices and will not be repeated here.

*3.2. Characterization of near causal reversed-frame-normalizing controllers*

From (10a) it can be shown that the  $p, q$  element of the RFNC is given by

$$k_{p,q}(z) = \sum_{i=1}^m y_{p,i}(z) u_{i,q}^*(z) k_i(z) \quad (16)$$

where  $y_{p,i}(z)$  and  $u_{i,q}(z)$  denote the  $p, i$  and  $i, q$  elements of  $Y(z)$  and  $U(z)$  respectively. Using the bicausal expansions of  $U(z)$  and  $Y(z)$  of Algorithm 3.1 and defining finite Laurent expansions for  $k_i(z)$ , we may derive an expression for the bicausal expansion

of  $k_{p,q}(z)$ . Let  $\mathbf{k}_{p,q}$  denote the vector of coefficients of the bicausal expansion of  $k_{p,q}(z)$  (taken in descending order of powers of  $z$ ). Then (16) implies that

$$\mathbf{k}_{p,q} = \sum_{i=1}^m C_{y_{pi}} C_{u_{iq}^*} \mathbf{k}_i \quad (17)$$

where  $\mathbf{k}_i$  is the vector of coefficients in the bicausal expansion of  $k_i(z)$  (again taken in descending powers of  $z$ ), and  $C_{y_{pi}}$ ,  $C_{u_{iq}^*}$  are lower triangular striped Toeplitz convolution matrices formed out of the coefficients of the bicausal expansions of  $y_{p,i}(z)$  and  $u_{i,q}^*(z)$  respectively. In particular, assuming for simplicity and without loss of generality that  $k_i(z)$  share a common causal and anticausal length with  $U(z)$  and  $Y(z)$ , then  $C_{y_{pi}} \in R^{(6\mu+1) \times (4\mu+1)}$ ,  $C_{u_{iq}^*} \in R^{(4\mu+1) \times (2\mu+1)}$  and the  $i, j$  element ( $j \leq i$ ) of these matrices is given by the  $(i-j)+1$  element of the sequence of coefficients in the respective bicausal expansion of  $y_{p,i}(z)$  and  $u_{i,q}^*(z)$  (taken in order of descending powers of  $z$ ) or zero if  $(i-j)+1 > 2\mu+1$ . It is easy to show that the first  $3\mu$  elements of  $\mathbf{k}_{p,q}$  correspond to the coefficients of the anticausal component in the Laurent expansion of  $k_{p,q}(z)$ , and this must be zero if  $k_{p,q}(z)$  is to be causal.

**Theorem 3.1:** Let  $C_{yu_{\tilde{p}i q}^+}$  denote the matrix formed out of the first  $3\mu$  rows of  $C_{y_{pi}} C_{u_{iq}^*}$  and let  $\mathbf{k} = [\mathbf{k}_1^T, \mathbf{k}_2^T, \dots, \mathbf{k}_m^T]^T$ , then the RFNC of (10a) will be causal if and only if

$$M\mathbf{k} = \mathbf{0}_{3\mu m} \quad (18a)$$

$$M = \begin{bmatrix} C_{yu_{\tilde{p}11}^+} & C_{yu_{\tilde{p}12}^+} & \dots & C_{yu_{\tilde{p}1m}^+} \\ C_{yu_{\tilde{p}21}^+} & C_{yu_{\tilde{p}22}^+} & \dots & C_{yu_{\tilde{p}2m}^+} \\ \vdots & \vdots & & \vdots \\ C_{yu_{\tilde{p}j1}^+} & C_{yu_{\tilde{p}j2}^+} & \dots & C_{yu_{\tilde{p}jm}^+} \\ C_{yu_{\tilde{p}12}^+} & C_{yu_{\tilde{p}122}^+} & \dots & C_{yu_{\tilde{p}1m_2}^+} \\ \vdots & \vdots & & \vdots \\ C_{yu_{\tilde{p}jm}^+} & C_{yu_{\tilde{p}2m}^+} & \dots & C_{yu_{\tilde{p}mm}^+} \end{bmatrix} \quad (18b)$$

**Proof:** This is a direct consequence of (17). □

**Corollary 3.1:** The matrix  $M$  of Theorem 3.1 always possesses a kernel.

**Proof:** If  $G(z)$  is the minimum phase, then  $N^{-1}(z) = Y(z)\Gamma_N^{-1}(z)U^*(z)$  will admit a causal power series expansion. Thus choosing  $\Gamma_K(z) = \Gamma_N^{-1}(z)$  in (10a) would result in a causal RFNC; clearly then the composite vector  $\mathbf{k}$  of the coefficients in the bicausal expansions of the diagonal elements of  $\Gamma_N^{-1}(z)$  will satisfy (18a) and will thus lie in the kernel of  $M$ . The same arguments also apply in the general case (when  $G(z)$  may not be the minimum phase) after  $N(z)$  has been scaled appropriately so as to ensure that  $N^{-1}(z)$  admits a causal expansion. □

The dimension of the kernel of  $M$  can be determined by the number of its zero singular values; however, in practice (owing to truncation errors) these will be small but not exactly zero. Thus a practical way to define a matrix representation  $Y_0$  for the kernel of  $M$  is as follows.

- (i) Reorder the singular value decomposition of  $M$  such that its singular values appear in non-increasing order.
- (ii) Let  $\nu$  be the number of singular values which are less than or equal to a small threshold value  $\epsilon$ .

- (iii) Form the matrix  $Y_0$  comprising as its column vectors the last  $\nu$  input principal directions of  $M$ .

Then any  $\mathbf{k}$  written as a linear combination of the columns of  $Y_0$  will satisfy the condition

$$\|M\mathbf{k}\| = \|X_0 \Sigma_0 \alpha\| \leq \varepsilon \|\alpha\| \quad \text{for } \mathbf{k} = Y_0 \alpha \quad (19)$$

and so for sufficiently large  $\mu$  and sufficiently small  $\varepsilon$  such  $\mathbf{k}$  will provide a nearly exact solution to (18a);  $X_0$  comprises the last  $\nu$  output principal directions of  $M$  whereas  $\Sigma_0$  is the diagonal matrices whose diagonal elements are given by the last  $\nu$  singular values of  $M$ .

**Remark 3.2:** In practice, it is often the case that the singular values of  $M$  do not group themselves naturally into two sets, according to whether their size is insignificant or not; the transition from significant to insignificant singular values is not abrupt. Thus the choice of the ‘correct’ value for  $\nu$  can be ambiguous. It is noted that the larger the value chosen for  $\nu$ , the more degrees of freedom are available for the choice of  $\mathbf{k}$ ; however, larger  $\nu$  imply larger  $\varepsilon$  and this results in a larger (but hopefully still insignificant) anticausal component in the RFNC of (10a). For the purposes of implementation, the anticausal component of  $K$  must be set equal to zero and the error associated with this truncation will affect the degree of reversed-frame normalization provided by  $K$ .

### 3.3. Design algorithm

Equations (9) and (10) combine to give

$$Q(z) = G(z)K(z) = U(z)\Gamma_Q(z)U^*(z), \quad \Gamma_Q(z) = \text{diag}\{q_i(z)\}, \quad q_i(z) = \gamma_i(z)k_i(z), \\ i = 1, \dots, m \quad (20)$$

where it is seen that the compensated transfer function matrix is normal (as intended) and the  $q_i(z)$  define its eigenvalue functions since  $U^*(z) = U^{-1}(z)$ . Clearly then the frequency response of the  $q_i(z)$  define the characteristic loci of  $Q(z)$  and hence determine the stability properties of the closed-loop system. The objective of  $K(z)$  therefore, over and above the achievement of normality, is to shape the gain and phase characteristics of  $q_i(z)$ .

To explore the relationship of  $q_i(z)$  to  $k_i(z)$  or indeed to the vector of degrees of freedom  $\alpha$  we may write

$$q_i(z) = \gamma_i(z)\phi(z)k_i, \quad \phi(z) = [z^\mu, z^{\mu-1}, \dots, z^{-\nu}] \quad (21)$$

from which we have

$$\mathbf{q}(z) = \Gamma(z)\Phi(z)\mathbf{k} = \Gamma(z)\Phi(z)Y_0 \alpha, \quad \Phi(z) = \text{diag}\{\phi^T(z), \phi^T(z), \dots, \phi^T(z)\} \quad (22)$$

Then invoking this condition at pre-selected points  $z_i, i = 0, 1, \dots, n-1$ , on the unit circle, say  $z_i = \exp\{ji2\pi/n\}$ , we get

$$\hat{\mathbf{q}} = \Psi \alpha \quad (23a)$$

$$\Phi = \begin{bmatrix} \Gamma(z_0)\Phi(z_0)Y_0 \\ \Gamma(z_1)\Phi(z_1)Y_0 \\ \vdots \\ \Gamma(z_{n-1})\Phi(z_{n-1})Y_0 \end{bmatrix} \quad (23b)$$



**Remark 3.3:** Equation (23) highlights the dependence of the achievable frequency response for  $q_f(z)$  on  $\alpha$  and suggests a systematic approach for the design of  $\alpha$ : choose target transfer functions for the  $q_f(z)$  and enforce (23) at the pre-selected set of frequencies. Given that the dimension of  $\alpha$  is  $\nu$  it follows that, in general, (23) cannot hold true for  $n > \nu/2$ , since  $\alpha$  is real and  $\hat{q}$  is complex. However, to avoid aliasing, and hence undesirable frequency intersample behaviour,  $n > \mu_q + 1$ , where  $\mu_q$  is defined in the same way as  $\mu$  but is based on the inverse Fourier transforms of the targets  $q_f(z)$  rather than the quasi-Nyquist decomposition of  $N(z)$ . The implication of this is that (23) should have more than  $2m(\mu_q + 1)$  equations, and in general this will be larger than the number of available degrees of freedom  $\nu$ ; hence, for a general set of  $q_f(z)$ , (23) will not have an exact solution. An alternative statement of this aspect is that target frequency responses for  $q_f(z)$  cannot be defined arbitrarily and indeed the totality of achievable frequency responses is defined by the range space of  $\Psi$ .  $\square$

The relative error between achievable and desired frequency responses given by

$$\eta_l(z) = \frac{q_f(z) - \gamma_l(z)\phi^T(z)Y_0\alpha}{q_f(z)} = 1 - \frac{\gamma_l(z)\phi^T(z)Y_0\alpha}{q_f(z)}, \quad l = 0, 1, \dots, n-1 \quad (24)$$

suggests that a sensible way to choose  $\alpha$  is offered by the following minimization problem:

$$\min_{\alpha} J_{\alpha} = \min_{\alpha} \|\mathbf{I} - \Xi\alpha\| \quad (25a)$$

$$\Xi = \begin{bmatrix} \Gamma_Q^{-1}(z_0)\Gamma(z_0)\Phi(z_0)Y_0 \\ \Gamma_Q^{-1}(z_1)\Gamma(z_1)\Phi(z_1)Y_0 \\ \vdots \\ \Gamma_Q^{-1}(z_{n-1})\Gamma(z_{n-1})\Phi(z_{n-1})Y_0 \end{bmatrix} \quad (25b)$$

where  $\mathbf{I}$  is a vector of ones of conformal dimension, and  $\Gamma_Q(z) = \text{diag}\{q_1(z), \dots, q_m(z)\}$ . Then the optimal choice of  $\alpha$  is as given below.

**Theorem 3.2:** *The vector  $\mathbf{k}$  of coefficients for the bicausal expansions of the quasi-Nyquist functions  $k_f(z)$  of the RFNC  $K(z)$  of (10a) which minimizes the euclidean norm of the vector of relative errors between desired characteristic loci for  $G(z)K(z)$  and achievable frequency responses evaluated at  $z_i = \exp\{ji2\pi/n\}$ ,  $i = 0, 1, \dots, n-1$  is given by*

$$\mathbf{k} = Y_0\alpha = Y_0\Re(\Xi^*\Xi)\Re(\Xi^*\mathbf{I}) \quad (26)$$

where  $\Xi$  is as defined in (25b).

**Proof:** This follows from the stationarity conditions of the minimization problem (25a).  $\square$

Apart from suggesting an optimal solution for  $\alpha$ , Theorem 3.2 also gives a measure of the ‘achievability’ of the chosen characteristic loci targets. The difficulty here lies in the fact that  $K(z)$  must not only achieve these targets but also must aim at normality of  $G(z)K(z)$  and  $K(z)G(z)$ . Herein can be found the contribution of Theorems 3.1 and 3.2; the former characterizes the restrictions placed on  $K(z)$  on account of the requirement for normality whereas the latter gives a measure of how close to targets one can get, given the restrictions imposed on  $K(z)$ . These two theorems suggest a systematic way for reaching a compromise between the two conflicting objectives of

normality and target attainment; the size of the threshold value of  $\varepsilon$  controls the accuracy with which  $K(z)$  achieves normality, whereas the accuracy of target attainment is controlled by how large  $\nu$  is chosen to be. The conflicting nature of the two objectives is exposed by the fact that target attainment in general requires large  $\nu$  whereas normality requires  $\varepsilon$  to be small; yet clearly small  $\varepsilon$  normally implies small  $\nu$ . Thus, for example, in cases where the uncertainty in  $G(z)$  is large and the requirement for normality is paramount, Theorem 3.1 suggests that  $\nu$  must be chosen to be small, and then Theorem 3.2 provides guidance with respect to the definition of achievable targets. Given that in general the choice of targets is by no means unique, guidance of this sort is clearly of practical importance.

#### 4. Design study

Consider the continuous-time linear model of an automobile gas turbine and the target transfer functions studied in Hung and MacFarlane (1982) which through the use of the bilinear transform  $s = (z - 1)/(z + 1)$  gives the frequency response equivalent discrete-time model:

$$G(z) = \frac{1}{d(z)} N(z) = \frac{1}{d(z)} \begin{bmatrix} n_{11}(z) & n_{12}(z) \\ n_{21}(z) & n_{22}(z) \end{bmatrix}$$

$$q_1(z) = \frac{5(1 + z^{-1})}{1 - z^{-1}}$$

$$q_2(z) = \frac{4.5455(1 + z^{-1})^2}{(1 - z^{-1})(1 + 0.8182z^{-1})}$$

$$\mathbf{n}_{11} = [1.4652 \ 4.0590 \ 2.4925 \ -2.6178 \ -4.1190 \ -1.4851 \ 1.0928 \ 1.4590 \ 0.2998 \\ -0.3030 \ -0.1104 \ 0.0087 \ -0.0002]$$

$$\mathbf{n}_{12} = [0.2690 \ 1.1520 \ 2.0358 \ 1.4559 \ -0.3837 \ -1.2588 \ -0.6696 \ -0.0409 \ 0.1308 \\ 0.1002 \ 0.0250 \ -0.0021]$$

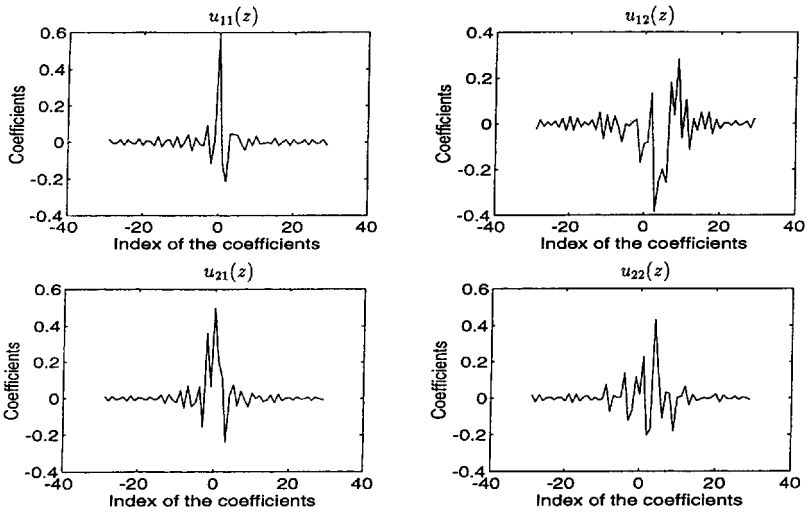
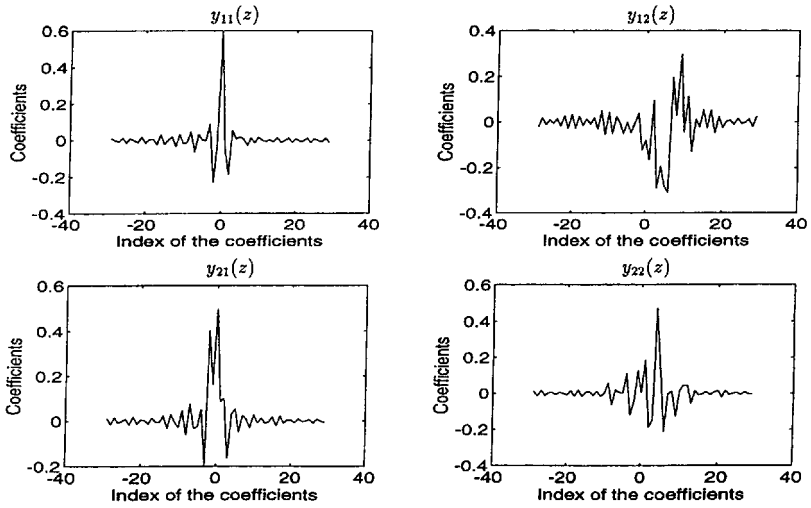
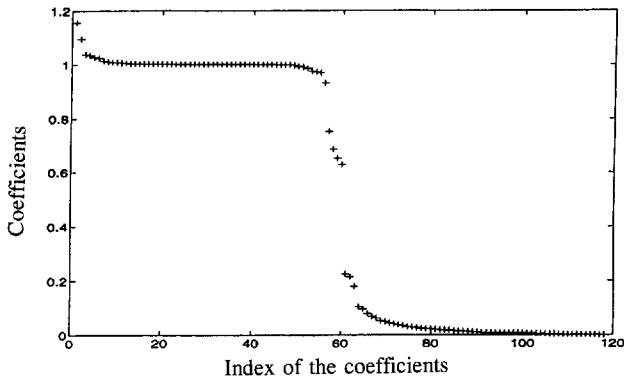
$$\mathbf{n}_{21} = [0.4100 \ 1.4219 \ 2.1296 \ 1.5818 \ 0.0118 \ -1.1266 \ -0.9928 \ -0.2966 \ 0.1469 \\ 0.1697 \ 0.0411 \ -0.0034 \ 0.0001]$$

$$\mathbf{n}_{22} = [2.0165 \ 6.1789 \ 7.0692 \ 3.2754 \ -1.2009 \ -3.5151 \ -2.7039 \ -0.5543 \ 0.5359 \\ 0.4359 \ 0.0955 \ -0.0082 \ 0.0002]$$

$$\mathbf{d} = [1 \ 1.7084 \ -0.0163 \ -1.4869 \ -0.9799 \ 0.0265 \ 0.5576 \ 0.2674 \ -0.1067 \\ -0.0600 \ 0.0069 \ -0.0002 \ 0.000003]$$

It is noted that the above targets are not analytic on the unit circle and thus cannot be represented by the Laurent expansions proposed earlier; to overcome this difficulty the  $1 - z^{-1}$  term present in  $q_1(z)$  and  $q_2(z)$  will be removed from the targets and will be included in  $d(z)$  instead.

The bicausal expansions for the  $U(z)$  and  $Y(z)$  of this model, computed after suitable diagonal unitary scaling, are shown in Figs. 1 and 2 and the 118 singular values of the corresponding  $M$  matrix of (18) are shown in Fig. 3 from which it is seen that  $\nu$  should be chosen to be 58 or less. If reversed-frame alignment and normality are a high priority, however, then  $\nu$  should be chosen to be considerably less. Thus setting  $\varepsilon = 0.002$  it is found that  $\nu = 4$  (the 114th and 115th singular values of  $M$  are 0.0021

Figure 1. Coefficients of the bicausal expansion of  $U(z)$ .Figure 2. Coefficients of the bicausal expansion of  $Y(z)$ .Figure 3. Singular values of  $M$ .

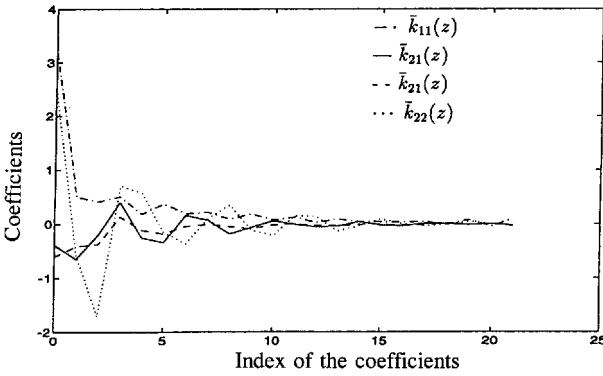


Figure 4. Coefficients of the polynomials of the numerator matrix of the controller transfer function.

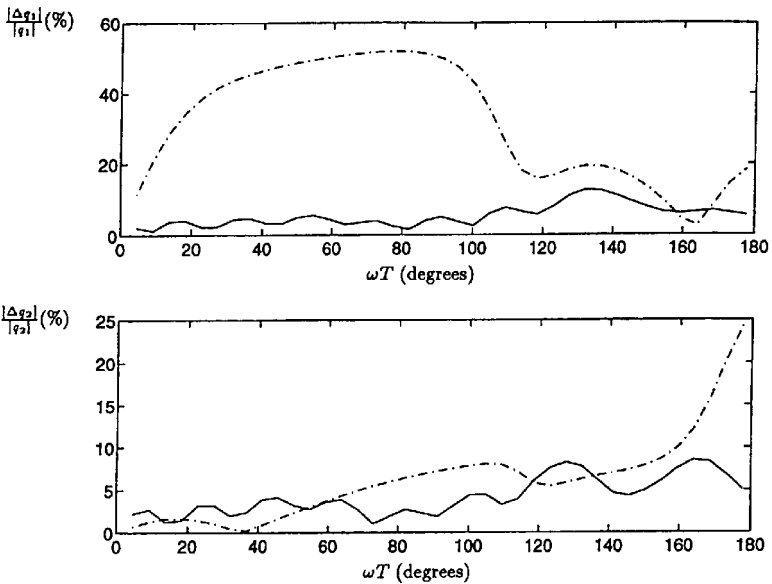


Figure 5. Percentage error between desired and achieved characteristic loci of  $Q(z)$ : (---), from Hung and MacFarlane (1982); (—), causal RFNC.

and 0.0012) and for this  $\nu$  the optimal choice for  $\alpha$  of Theorem 3.2 result in a RFNC which gives a  $Q(z) = G(z)K(z)$  with  $J_{X,Y} < 0.008$  (for all frequencies) and  $J_\alpha$  is 7.7428. The implication of these figures is that such a  $K(z)$  attains near normality (as measured by  $J_{X,Y}$ ) but gives very poor target attainment. In order to improve the latter,  $\nu$  is increased to 58 and  $K(z)$  is recomputed; the causal sequence expansion of this  $K(z)$  is shown in Fig. 4. As expected, the increase in  $\nu$  results in much better target attainment (the new value for  $J_\alpha$  is 0.8984) but this is achieved at the cost of normality (the new value for  $J_{X,Y} \leq 0.15$  and in fact  $J_{X,Y} < 0.03$  for all  $\omega$  such that  $\omega T < 2\pi/3$ ). It is interesting to note that this controller outperforms the better of the two controllers proposed by Hung and MacFarlane (obtained by the nonlinear optimization problem described in §2) in both respects (normality and target attainment) as illustrated in Figs 5 and 6. For completeness, the design study is concluded by performing some closed-loop time-response simulations, the results of which are shown in Fig. 7; Fig. 7(a) shows the response of outputs 1 and 2 to a unit step demand on output 1, whereas

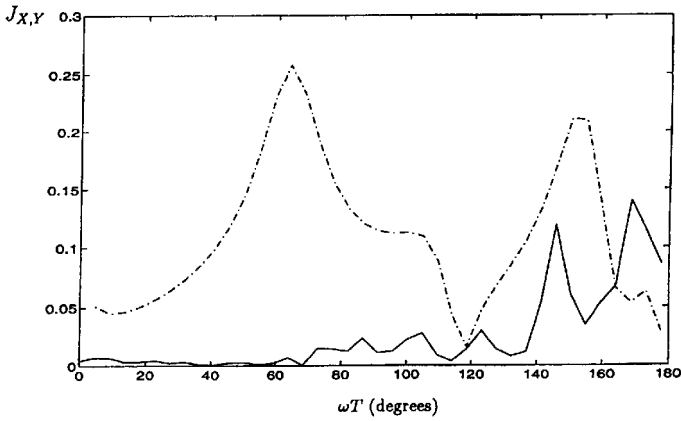


Figure 6. Value of  $J_{X,Y}$  for  $Q(z) = G(z)K(z)$ ; (---), from Hung and MacFarlane (1982); (—), causal RFNC.

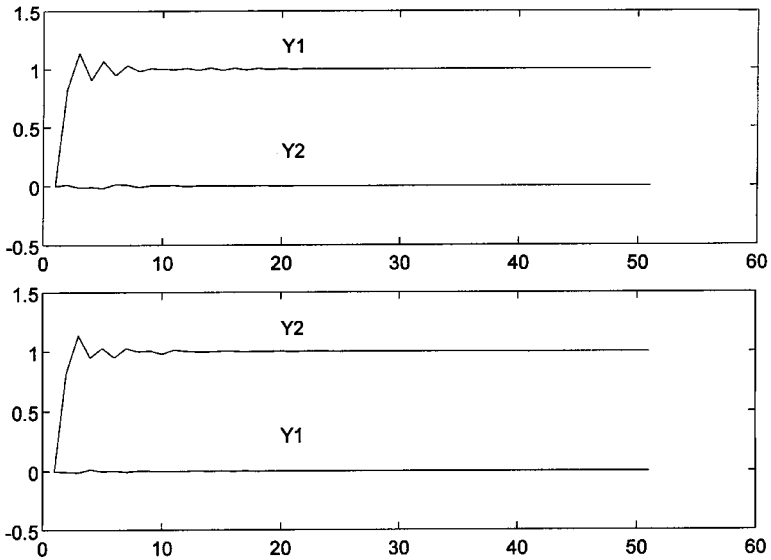


Figure 7. Closed-loop step responses of outputs 1 and 2 to (a) the reference  $[1, 0]$  and (b) the reference  $[0, 1]$ , where calibration of the  $x$  axes corresponds to the number of sampling instants.

Fig. 7(b) shows the corresponding responses to a unit step demand on output 2. Clearly the responses are satisfactory; they are fast, displaying no steady-state error and only a limited amount of overshoot, and they are largely non-interactive.

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