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The use of rational eigenvector approximations in commutative controllers

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Commutative controllers provide a convenient means for the systematic adjustment of the gain/phase characteristic of generalized Nyquist diagrams. Due to the irrational nature of the eigenvectors of transfer function matrices, practical commutative controllers are based on eigenvector approximations. This paper investigates the use of rational approximations, highlights stability difficulties that arise in connection with unstable fixed modes and derives the necessary and sufficient conditions that overcome these difficulties. The results of the paper are illustrated by means of numerical examples.

1. Introduction

Consider the discrete time feedback system of Fig. 1 in which $G(z)$, and $K(z)$ are the $m \times m$ rational transfer function matrices of the model of a multivariable system and a controller, respectively. According to the generalized Nyquist criterion (MacFarlane and Postlethwaite 1977), the closed-loop system is stable if, and only if, the net sum of counterclockwise critical point encirclements by the characteristic loci (CL) of the open-loop transfer function matrix $Q(z) = G(z)K(z)$ is equal to the number of unstable poles of $G(z)$ and $K(z)$. The CL of $Q(z)$ are defined as the frequency responses of the eigenfunctions $q_i(z)$ for $i = 1, 2, \ldots, m$, of $Q(z)$, which are defined as the branches of characteristic functions of $Q(z)$; it is pointed out that in some cases the CL have to be joined together in order to form closed curves. Furthermore, by arguments of analyticity and conformal mapping it is possible to establish a connection between the relative stability margins exhibited by the individual CL of $Q(z)$ and the positions of the closed loop poles. Thus, the manipulation of the CL of $Q(z)$ forms an obvious objective in the design of multivariable controllers.

In this context the design problem can be stated as follows: if the CL of $G(z)$ have undesirable gain/phase characteristics, derive a $K(z)$ which will, in a systematic way, modify these and thus yield a $Q(z)$ which satisfies the generalized Nyquist criterion with adequate stability margins. An elegant

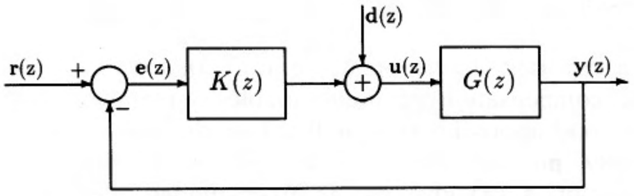

Figure 1. Block diagram for the design of internally stabilizing controllers.

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theoretical solution to this problem is provided by the concept of commutativity (MacFarlane and Belletrutti 1973): if $K(z)$ is such that $G(z)K(z) = K(z)G(z)$, then $G(z)$ and $K(z)$ will share a common eigenvector matrix $W(z)$ and the corresponding eigenfunctions of $G(z)$, $g_i(z)$, and of $K(z)$, $k_i(z)$, will multiply to give the eigenfunctions of $Q(z)$ as $q_i(z) = g_i(z)k_i(z)$, for $i = 1, 2, \ldots, m$. Unfortunately, in general, the eigenvector functions of $G(z)$ are irrational functions of z, and thus commutative controllers do not offer a practicable solution. An obvious way around this difficulty is to define $K(z)$ to have $W_a(z)$, $V_a(z)$ as its right and left eigenvector matrices, where $W_a(z)$, $V_a(z)$ can be chosen to be 'good' approximations to $W(z)$, $V(z) = W(z)^{-1}$, respectively. To keep the controller simple one could set $V_a(z) = W_a(z)^{-1}$ and choose $W_a(z)$ to be a constant real matrix selected to give the 'best' approximation to $W(z)$ at one frequency only (MacFarlane and Kouvaritakis 1977); such a commutative controller however would only give approximate commutativity over a limited band of frequencies. Alternatively one could choose $W_a(z)$ and $V_a(z)$ to be causal power series approximations to $W(z)$ and $V(z)$ (Cloud and Kouvaritakis 1987). The resulting controller can be made to be commutative to within any arbitrary degree for the case where $G(z)$ does not have unstable branch points. To handle the general case, a recent paper (Kouvaritakis and Basilio 1994) proposed the use of bicausal power series approximations for $W(z)$, $V(z)$ as well as for the eigenvalue matrix, $\Lambda_K(z)$, of the controller itself; it was then shown that by constraining the degrees of freedom available in $\Lambda_K(z)$ it is possible to design a controller which is causal and commutes with $G(z)$ on the unit circle to within any desired degree of accuracy. The remaining degrees of freedom available in $\Lambda_K(z)$ can be deployed for the purpose of adjusting the gain/phase characteristics of the CL of $G(z)$. We shall refer to this controller as a causal commutative controller, CCC for short.

The need for causality implies that $\Lambda_K(z)$ cannot be chosen arbitrarily so that CCC cannot achieve exactly any target CL for $Q(z)$. To overcome this problem the bicausal representation of $\Lambda_K(z)$ can be chosen so as to minimize a cost which places a penalty on the deviation of the CL of $Q(z)$ from target values. Given the degrees of freedom available in the selection of target CL, it is important to have the means of specifying 'reasonable' targets, namely targets which result in small costs. In an attempt to resolve this issue, in this paper we consider the use of $W_a(z) = W^{\#}(z)$ and $V_a(z) = V^{\#}(z) = W^{\#}(z)^{-1}$, with $W^{\#}(z)$ rational, and determine necessary and sufficient conditions for causality which, in this context, boils down to an issue of stability. For convenience we shall refer to the resulting controller as a rational commutative controller, RCC for short.

It has been reported in the open literature that RCCs provide practicable means for the compensation of multivariable systems, so long of course as $W^{\#}(z)$ gives a good approximation of $W(z)$ on the unit circle (see for example Maciejowski 1989, pp. 145–146). It is the purpose of this paper to show that arbitrary choices of $\Lambda_K(z)$ give rise to fixed modes, namely poles which are invariant under the use of scalar feedback. Under exact commutativity such fixed modes are shown to be fixed modes of $Q(z)$ which can therefore lead to instability. It is further argued that under near commutativity the controller fixed modes give rise to 'near fixed' modes in $Q(z)$ which are nearly invariant under scalar feedback in the sense that appreciable changes in them require the use of arbitrarily large scalar gains. The paper gives careful consideration to the issue of internal stability, and determines the necessary and sufficient condition for the elimination of undesirable fixed modes.

2. Rational eigenvector approximations

Let $d(z)$ denote the least common denominator of $G(z)$, and define the numerator matrix polynomial of $G(z)$ to be $N(z) = d(z)G(z)$. Then, almost everywhere, except at branch points, the eigenfunctions $g_i(z)$, of $G(z)$ form distinct analytic functions given as $g_i(z) = n_i(z)/d(z)$, where $n_i(z)$ are the branches of the algebraic equation

$$\det [n(z)I_m - N(z)] = 0 \tag{1}$$

The branch points in question are the points at which $n(z)$ is not differentiable. For brevity, branch points henceforth will be referred to as BPs. It is known (Kouvaritakis *et al*. 1990, Kouvaritakis and Rossiter 1991) that under the assumption that the unit circle does not go through any BPs and does not cross any branch cuts, the $n_i(z)$ and the corresponding right and left eigenvectors, $w_i(z)$ and $v_i(z)$, respectively admit Laurent expansions of the form

$$n_i(z) = \sum_{k=-\infty}^{\infty} n_{i_k} z^{-k}, \quad w_i(z) = \sum_{k=-\infty}^{\infty} w_{i_k} z^{-k}, \quad v_i(z) = \sum_{k=-\infty}^{\infty} v_{i_k} z^{-k} \tag{2}$$

which are valid everywhere inside an annulus containing the unit circle. It is assumed that the $v_i(z)$ are scaled so that $v_i^{\mathrm{T}}(z)w_i(z) = 1$. Furthermore, the sequences of coefficients for positive k (referred to as causal sequences) and the sequences for negative k (anti-causal sequences) all converge to zero. Hence, for practical purposes, (2) can be replaced by

$$n_i(z) = \sum_{k=-\mu}^{\mu} n_{i_k} z^{-k} \tag{3 a}$$

$$w_i(z) = \sum_{k=-\mu}^{\mu} w_{i_k} z^{-k} \tag{3 b}$$

$$v_{i_k}(z) = \sum_{k=-\mu}^{\mu} v_{i_k} z^{-k} \tag{3 c}$$

where μ denotes the number of significant strictly causal or anti-causal terms (whichever is greatest). The computation of these sequences can be carried out in a robust and efficient numerical way through a process of sampling the frequency response of $n_i(z)$, $w_i(z)$ and $v_i(z)$ and a process of inverse discrete Fourier transformation.

Equation (3) shows that it is possible to get rational $W^{\#}(z)$ which constitute good approximations to $W(z)$ on the unit circle. Note that due to the arbitrariness in the scaling of eigenvectors, without loss of generality, it is possible to assume that $W^{\#}(z)$ is a matrix polynomial in powers of z^{-1}; this assumption will be adopted throughout this paper. An obvious good choice for $W^{\#}(z)$ is therefore provided by (3 b): each column of $W^{\#}(z)$ can be chosen to be the corresponding right-hand side of (3 b) after premultiplication of $z^{-\mu}$. However, the resulting $W^{\#}(z)$ and $V^{\#}(z) = W^{\#}(z)^{-1}$ would involve powers of up to $z^{-2\mu}$ and would result in unacceptably high order RCCs. Here we shall restrict the degree of the ith column of $W^{\#}(z)$, $w_i^{\#}(z)$, to be v_i and shall be looking for solutions which minimize some measure of the $\phi_i(\omega)$ of the misalignment between $w_i[\exp(j\omega T)]$ and $w_i^{\#}[\exp(j\omega T)]$; T denotes the sampling interval. In this context the optimal solution can be defined as the $w_i^{\#}(z)$ which minimizes $\|\phi_i\|_\infty = \max_\omega |\phi_i(\omega)|$.

A simple way to deal with the problem of eigenvector approximation is to adopt an element-by-element approach; according to this, one selects the p, q element of $W^{\#}(z)$ so as to minimize $\|\varepsilon_{pq}\|_\infty$ where $\varepsilon_{pq}(\omega) = |w_{pq}[\exp(j\omega T)] - w_{pq}^{\#}[\exp(j\omega t)]|$. Using the Caratheodory/Fejer theory (Caratheodory and Fejer, 1911) it is possible to state that an optimal solution exists, is unique, and results in a 'circular' (all-pass) error function. Unfortunately, this solution in general will be of high order, so one can invoke Trefethen's work (Trefethen 1981), according to which 'near-circularity' implies near optimality, to get approximations of order v_q. For large v_q, both $\|\varepsilon_{pq}\|_\infty$ and hence $\|\phi_q\|_\infty$ will be sufficiently small so that the fact that one is minimizing the first norm for $p = 1, 2, \ldots, m$ instead of the second will not matter. However, for small v_q this solution will be sub-optimal, because the approach does not take explicit account of the vectorial nature of the problem. Thus, for small v_q a different approach is needed and one such was proposed by Kouvaritakis *et al.* (1990). According to this, $\phi_i(\omega)$ is taken to be the euclidean norm of the error incurred in the identity $V[\exp(j\omega T)]w_i[\exp(j\omega T)] = e_i$ when $w_i[\exp(j\omega T)]$ is replaced by $w_i^{\#}[\exp(j\omega T)]$; e_i denotes the ith column of the identity matrix I_m of dimension m. In particular, $\phi_i(\omega)$ is defined to be

$$\phi_i(\omega) = \|V[e^{j\omega T}]w_i^{\#}[e^{j\omega T}] - e_i\|_2; \quad i = 1, 2, \ldots, m \tag{4}$$

The optimal solution can then be obtained through the use of an extension of Lawson's weighted least squares algorithm (Lawson 1961) according to which $\phi_i(\omega)$ is made as near-circular as possible. To ensure uniformity over all frequencies the eigenvectors are normalized so that $\|w_i[\exp(j\omega T)]\|_2 = 1$, but that still allows some further degrees of freedom since the eigenvectors can still be scaled with a unitary factor of the form $\exp(j\psi_i(\omega))$. For a given $w_i^{\#}(z)$ the cost $\|\phi_i\|_\infty$ can be reduced further if the phases of the scaling factors are given their optimal values, namely the values which minimize $|\phi_i(\omega)|$

$$\psi_i(\omega) = -\text{phase}\,\{v_i^{\mathrm{T}}[\exp(j\omega T)]w_i^{\#}[\exp(j\omega T)]\} \tag{5}$$

The overall algorithm for the computation of eigenvector approximates can be as follows.

Step 1. Select a set of frequencies such that $\exp(j\omega T)$ are evenly distributed around the unit circle; the number of frequencies should be taken to be large, and in any case should be larger than 2μ.

Step 2. Scale the eigenvectors so as to have unity norm at each frequency point.

Step 3. Apply the weighted least squares algorithm to obtain the vector θ_i of the coefficients of $w_i^{\#}(z)$

$$\theta_i = [(w_i^{\#})_0^{\mathrm{T}}, (w_i^{\#})_1^{\mathrm{T}}, \ldots, (w_i^{\#})_{v_i}^{\mathrm{T}}]^{\mathrm{T}} \tag{6}$$

Step 4. Apply the unitary eigenvector scaling in accordance with (5).

Step 5. Repeat Steps 3–4 until the improvement in the cost falls below a preset threshold.

Remark 2.1: The convergence of the algorithm is not affected by the iterative application of Steps 3 and 4 because the optimal value of the cost is made less than or (in the limit equal to) the optimal value at the previous step. □

This algorithm works well, but here we improve upon it by the introduction of further degrees of freedom. In particular, we note that on account of the arbitrariness in the scaling of eigenvectors a more suitable measure of misalignment between $w_i[\exp(j\omega T)]$ and $w_i^{\#}[\exp(j\omega T)]$ than the measure of (4) is given by

$$\phi_i(\omega) = \|V[e^{j\omega T}]w_i^{\#}[e^{j\omega T}] - p_i[e^{j\omega T}]\,e_i\|_2; \quad i = 1, 2, \ldots, m \tag{7}$$

where $p_i(z)$ can be taken to be any arbitrary function with bi-causal expansion

$$p_i(z) = \sum_{-\mu}^{\mu} p_k^{(i)} z^{-k} \tag{8}$$

A convenient way to avoid the trivial solution is to set $p_0^{(i)}$ arbitrarily equal to 1, but otherwise the remaining coefficients $p_k^{(i)}$ for $k \neq 0$ represent degrees of freedom which can be taken together with the vector coefficients of $w_i^{\#}(z)$, $[w_i^{\#}]_k$, for $k = 1, 2, \ldots, v_i$ to form the overall vector of degrees of freedom

$$\theta_i = [(w_i^{\#})_0^{\mathrm{T}}, (w_i^{\#})_1^{\mathrm{T}}, \ldots, (w_i^{\#})_{v_i}^{\mathrm{T}}, p_{-\mu}^{(i)}, p_{-\mu+1}^{(i)}, \ldots, p_{\mu}^{(i)}]^{\mathrm{T}} \tag{9}$$

Then $\phi_i(\omega)$ can be written as

$$\begin{aligned} \phi_i(\omega) &= \|P(\omega)\theta_i - e_i\|_2; \\ P(\omega) &= \{V(e^{j\omega T})[I_m, e^{-j\omega T}I_m, \ldots, e^{-jv_i\omega T}I_m], \\ &\quad -e^{j\mu\omega T}e_i, \ldots, -e^{j\omega T}e_i, -e^{-j\omega T}e_i, \ldots, -e^{-j\mu\omega T}e_i\} \end{aligned} \tag{10}$$

This measure of misalignment suggests the following simple algorithm.

Algorithm 2.1:

Step 1. Select a set of frequencies, ω_k, $k = 1, 2, \ldots, N$ such that $\exp(j\omega_k T)$ are evenly distributed around the unit circle; N should be taken to be large, and in any case $N \geqslant 2\mu + 1$.

Step 2. Scale the eigenvectors so that at each frequency point they have unity norm.

Step 3. Initially define the set of weights (to be used in the step below) to be $w_k^{(0)} = 1/N$, for $k = 1, 2, \ldots, N$

Step 4. Compute the solution $\theta_i^{(j)}$ which minimizes the cost

$$J^{(j)} = \sum_{k=1}^{N} w_k^{(j)} \|P(\omega_k)\theta_i^{(j)} - e_i\|_2^2 \tag{11}$$

and update the weights in accordance with the recurrence relationship

$$w_k^{(j+1)} = \frac{w_k^{(j)} \|P(\omega_k)\theta_i^{(j)} - e_i\|_2}{\sum_{k=1}^{N} w_k^{(j)} \|P(\omega_k)\theta_i^{(j)} - e_i\|_2} \tag{12}$$

Step 5. Increment j and repeat Step 3 until the improvement in the cost falls below a preset threshold value.

The iterative procedure of Steps 3 and 4 constitutes an extension of the Lawson algorithm and is thus guaranteed to converge. Finally, on account of the extra degrees of freedom introduced through the use of $p_i(z)$, this algorithm will outperform the earlier algorithm of Kouvaritakis *et al.*, 1990; the numerical example below gives a clear illustration of this. Before discussing the example however, the following remark is in order.

Remark 2.2: Overparametrization of the eigenvector approximation problem will result in solutions $w_i^{\#}(z)$ whose elements will share some nearly coincidental zeros. Given the arbitrariness of eigenvector scaling these near common factors can be removed, and this will be assumed to be the case in the following. □

2.1. *Numerical example*

The right and left eigenvector matrices of the polynomial matrix

$$N(z) = \begin{bmatrix} -0{\cdot}2885 & -0{\cdot}3464 \\ 0{\cdot}4991 & 0{\cdot}1305 \end{bmatrix} + \begin{bmatrix} 0{\cdot}1164 & -0{\cdot}4991 \\ -0{\cdot}4994 & 0{\cdot}2734 \end{bmatrix} z^{-1} + \begin{bmatrix} 0{\cdot}2273 & -0{\cdot}0823 \\ -0{\cdot}1808 & 0{\cdot}1825 \end{bmatrix} z^{-2} \tag{13}$$

have the bicausal expansion shown in Fig. 2(a, b) from which it can be seen that μ is about 11. Hence, for this example, the minimum number of frequency points required for the application of Algorithm 2.1 is $2\mu + 1 = 23$; for improved accuracy we shall take $\mu = 24$ and $N = 60$. We shall look for fourth-order eigenvector approximations and thus we set $\nu_1 = \nu_2 = 4$. After ten cycles of Steps 3 and 4 of Algorithm 2.1, the measures $\|\phi_1\|_\infty$ and $\|\phi_2\|_\infty$ of misalignment between $w_i(z)$ and the resulting approximations $w_i^{\#}(z)$ were found to be 0·004 and 0·002, respectively. The corresponding eigenvector fourth order approximation was given by

$$W^{\#}(z) = \begin{bmatrix} 0{\cdot}1436 & -0{\cdot}0893 \\ -0{\cdot}1489 & -0{\cdot}046 \end{bmatrix} + \begin{bmatrix} 0{\cdot}3998 & -0{\cdot}5763 \\ 0{\cdot}5527 & 0{\cdot}8019 \end{bmatrix} z^{-1} + \begin{bmatrix} -1{\cdot}1154 & 0{\cdot}6484 \\ 1{\cdot}2525 & 0{\cdot}4334 \end{bmatrix} z^{-2} + \begin{bmatrix} -2{\cdot}0351 & 2{\cdot}5816 \\ -1{\cdot}7678 & -3{\cdot}0304 \end{bmatrix} z^{-3} + \begin{bmatrix} -0{\cdot}3354 & 0{\cdot}5809 \\ -0{\cdot}5984 & -0{\cdot}7172 \end{bmatrix} z^{-4} \tag{14}$$

and the actual misalignment angles between the eigenvectors and their respective approximations computed at the preselected frequency points are shown in Fig. 3(a). Clearly the algorithm produced an excellent result; at all frequencies

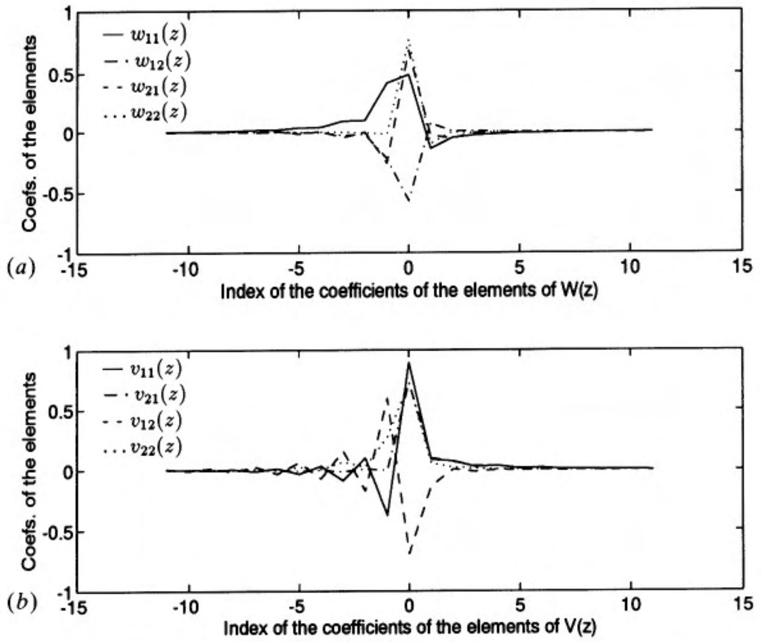


Figure 2. Bi-causal expansion of the elements of $W(z)$ and $V(z)$.

the misalignment angles are less than about 0·07°. The corresponding misalignment angles for the eigenvectors produced by the application of the earlier algorithm are shown in Fig. 3(*b*) and can be seen to be small, but in fact, are one order of magnitude larger than the angles of Fig. 3(*a*).

3. Fixed modes and internal stability

In § 2 we saw that is possible to get good rational eigenvector approximations which can be used in the construction of Rational Commutative Controllers of the form

$$K(z) = W^{\#}(z)\Lambda_{\mathrm{K}}(z)V^{\#}(z); \quad V^{\#}(z) = [W^{\#}(z)]^{-1}; \quad \Lambda_{\mathrm{K}}(z) = \mathrm{diag}\,[k_i(z)] \tag{15}$$

where $k_i(z)$ denotes rational transfer functions to be chosen by the designer for the purposes of adjusting the gain/phase characteristics of the CL of $G(z)$. The tacit assumption that is normally made in the literature is that the $k_i(z)$ can be chosen so that the CL of $Q(z) = G(z)K(z)$ satisfy the generalized Nyquist criterion, and this in turn guarantees closed-loop stability. In this section we take a close look at the issue of internal stability and highlight some difficulties that arise through the use of RCCs which relate to the existence of unstable fixed modes.

Definition 3.1: z_0 is a fixed mode (FM) of $G(z)$ if z_0 is a pole $G(z)$, and the sums of all the $i \times i$ principal minors of $G(z)$ are finite at z_0, for $i = 1, 2, \ldots, m$. □

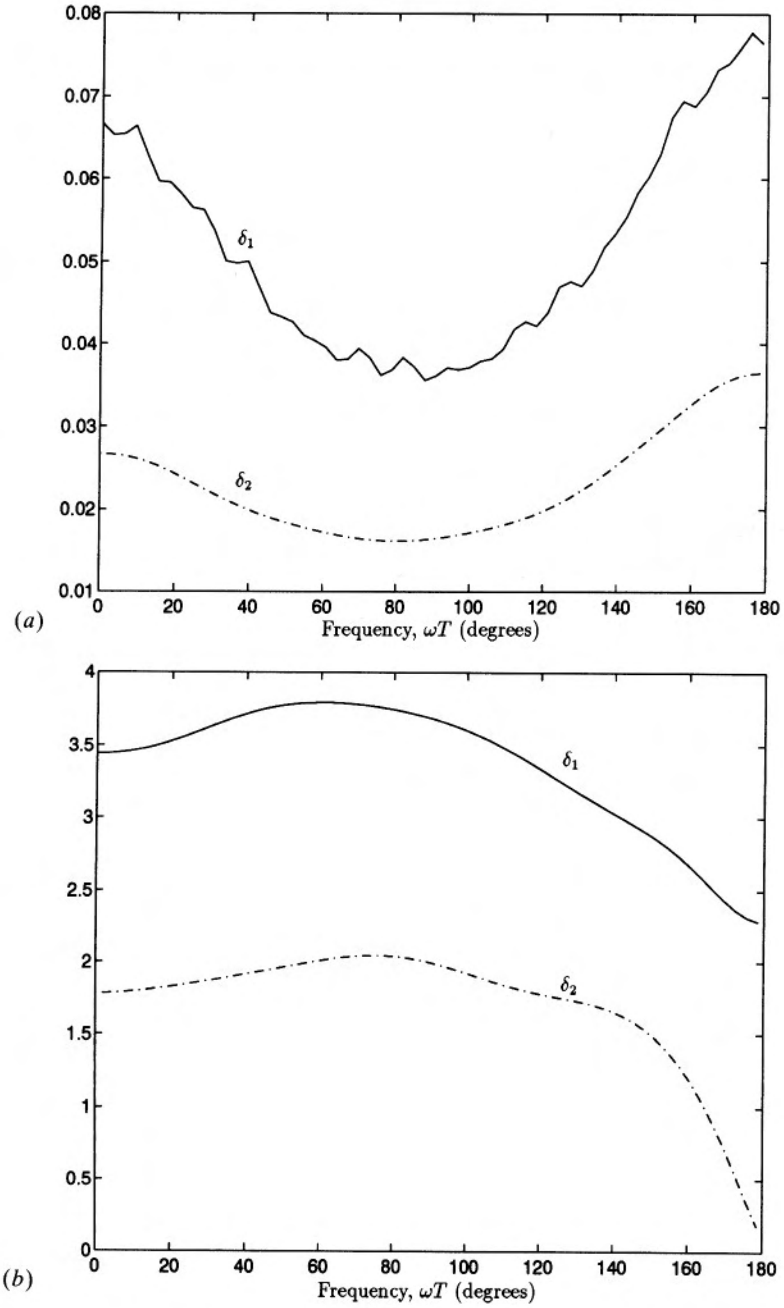


Figure 3. Misalignment angles (in degrees) between $W(z)$ and $W^{\#}(z)$ (a), and misalignment angles (in degrees) for $W(z)$ and $W^{\#}(z)$ (earlier algorithm) (b).

The implication of this definition is that a FM of $G(z)$, although being a pole of $G(z)$, is not a pole of any of its eigenfunctions. It is easy to show that such a pole remains invariant under the use of scalar feedback (i.e. when $K(z) = kI_{\mathrm{m}}$) for all values of scalar gain k; hence the terminology 'fixed mode'. Clearly, if the open-loop transfer function matrix $Q(z) = G(z)K(z)$ of Fig. 1 possessed unstable FM, then the closed-loop system would be unstable and would remain so even if a scalar gain other than unity were used around the loop.

We begin our discussion by considering the internal stability of the system of Fig. 1 for a general (not necessarily) commutative controller.

Theorem 3.1: *Assume that $G(z)$ and $K(z)$ have no decoupled modes. Then the feedback system of Fig. 1 is internally stable if, and only if:*

(*a*) *there are no unstable poles of $G(z)$ or $K(z)$ which are not poles of $Q(z) = G(z)K(z)$ (with the same multiplicity);*

(*b*) *the net sum of counter-clockwise encirclements of the critical point $(-1 + j0)$ by the CL of $Q(z)$ is equal to the number of unstable poles of $Q(z)$.*

Proof: Let $G(z) = D_G(z)^{-1}N_G(z)$ and $K(z) = N_K(z)D_K^{-1}(z)$ denote respectively a left and right coprime factorization of $G(z)$ and $K(z)$ so that

$$\left.\begin{aligned} M_1 &= (I + GK)^{-1} = D_K(N_GN_K + D_GD_K)^{-1}D_G; \\ M_2 &= (I + GK)^{-1}G = D_K(N_GN_K + D_GD_K)^{-1}N_G \\ M_3 &= K(I + GK)^{-1} = N_K(N_GN_K + D_GD_K)^{-1}D_G; \\ M_4 &= (I + KG)^{-1} = I - N_K(N_GN_K + D_GD_K)^{-1}N_G \end{aligned}\right\} \quad (16)$$

where, for convenience, we have dropped the argument z. We have internal stability if, and only if, M_i are stable for $i = 1, 2, 3, 4$.

Now let ψ, and d stand for characteristic, and cancelled pole polynomials respectively and let $\psi = pd$. Then the following relationships are easy to show

$$\left.\begin{aligned} &\det(D_G) = \psi_G = p_Gd_G; \quad \det(D_K) = \psi_K = p_Kd_K \\ &\det(N_GN_K + D_GD_K) = \psi_R = p_Rd_Gd_K \\ &\det(M_1) = \det(M_4) = \frac{p_Gp_K}{p_R}; \quad \det(M_2) = \frac{\det(N_G)p_K}{p_Rd_G} \\ &\det(M_3) = \frac{\det(N_K)p_G}{p_Rd_K} \end{aligned}\right\} \quad (17)$$

where R denotes the closed-loop transfer function matrix, d_G and d_K denote the polynomials whose roots are those poles of G and K (respectively) which do not appear in Q. The important point to note is that due to coprimeness, $\det(N_G)$ and d_G cannot have common roots, and the same applies to $\det(N_K)$ and d_K.

With (17) in mind it is easy to complete the proof of the theorem. Thus, take the necessity of conditions (*a*)–(*b*). If (*b*) were not true then p_R would not be Hurwitz and so all the M_i would be unstable. If, on the other hand, (*a*) did not hold then either d_G and/or d_K would not be Hurwitz and so M_2 and/or M_3 would be unstable. Conversely, if both (*a*), (*b*) conditions are true than p_R, d_G and d_K would be Hurwitz and all the M_i would be stable. □

Theorem 3.1 is a re-statement of the generalized Nyquist criterion with two differences: (i) it concerns 'internal' stability; and (ii) it makes no explicit mention of FM. In fact, the condition that Q should not have unstable FM which is included in the statement of the generalized Nyquist criterion is redundant because it is subsumed in condition (b) of Theorem 3.1, in the sense that if Q has unstable FM then the closed-loop system will be unstable and (b) cannot hold true, and conversely if (b) holds true then Q cannot have unstable FM. However, it should be clear that unstable FM, whether mentioned explicitly in the stability criterion or not, would lead to instability and thus form an important consideration in the choice of a controller K. The conditions for Q to have FMs are very strong, and require K to satisfy a very special set of conditions that will not be met by a general K. However, in the following we show that, in general, RCCs of the form given in (15) will have FMs which, under exact commutativity will also be the FMs of Q, and which under nearly exact commutativity will be 'near' FMs of Q thereby creating difficulties with respect to condition (b) of Theorem 3.1. It is therefore important to establish the conditions under which RCC's have FMs and this is undertaken in the next section.

4. Necessary and sufficient conditions for the non-existence of FMs

Definition 3.1 which concerns the FMs of $G(z)$ can also be applied to $K(z)$, however the physical meaning of such modes is somewhat artificial. In particular, if G, in Fig. 1, were replaced by $kI_{\rm m}$, then FMs of $K(z)$ would be poles of the resulting feedback system for all values of the scalar gain k.

Lemma 4.1: *z_0 can only be a FM of the RCC of* (15) *if z_0 is a zero of the determinant of $W^{\#}(z)$.*

Proof: FMs of K are poles of K which do not appear in its eigenfunctions, hence they must appear as poles of $V^{\#}(z)$; they cannot appear as poles of $W^{\#}(z)$ since this is polynomial in z. The result follows from the fact that $V^{\#}(z) = W^{\#}(z)^{-1} = \operatorname{adj}(W^{\#}(z))/\det(W^{\#}(z))$. □

Lemma 4.2: *Let the dyad $D_i(z)$ be defined as the product $w_i^{\#}(z)v_i^{\#}(z)^{\rm T}$, where $v_i^{\#}(z)^{\rm T}$ denotes the ith row of $V^{\#}(z)$. Then each zero of* $\det(W^{\#}(z))$ *appears as a pole of at least two dyads $D_i(z)$.*

Proof: By definition, at a zero of $\det(W^{\#}(z))$, z_0, at least one of the row vectors of $V^{\#}(z)$ must become unbounded, so that at least one of the dyads $D_i(z)$ must also become unbounded; this follows from Remark 2.2 according to which $w_i^{\#}(z_0)$ cannot be entirely zero. However, since $V^{\#}(z) = W^{\#}(z)^{-1}$, we have that

$$W^{\#}(z)V^{\#}(z) = D_1(z) + D_2(z) + \cdots + D_{\rm m}(z) = I_m; \quad D_i(z) = w_i^{\#}(z)v_i^{\#{\rm T}}(z) \tag{18}$$

But this cannot be true for z approaching z_0 unless at least two dyads become unbounded. □

Definition 4.1: Let z_0 be a zero of $\det(W^{\#}(z))$ and hence a pole of $V^{\#}(z)$. Then z_0 will be said to be a pole (of $V^{\#}(z)$) of geometric multiplicity ρ, if it appears in ρ invariant factors of $W^{\#}(z)$, and algebraic multiplity σ, if this is the highest power of all the elementary divisors of $W^{\#}(z)$ of the form $(z - z_0)^i$. □

Theorem 4.1: *Let z_0 be a pole of $V^{\#}(z)$ of algebraic and geometric multiplicity* 1, *and let $z - z_0$ appear in the denominator of $D_i(z)$ for $i \in I$, where I is a subset* *(not necessarily a proper subset) of the integers* $1, 2, \ldots, m$. *Assume also that* z_0 *is not a pole of any of the eigenfunctions* $k_i(z)$ *of the RCC of* (15). *Then* z_0 *will not be a FM of* $K(z)$, *if, and only if*

$$k_i(z) - k_j(z) = p(z)\phi_{ij}(z), \quad \forall\, i, j \in I; \quad p(z) = z - z_0 \tag{19}$$

where $\phi_{ij}(z)$ *is a rational function whose denominator is not a multiple of* $p(z)$.

Proof: Let z_0 appear in ρ dyads D_i, and let the dyads be re-ordered so that z_0 appears as a pole of the last ρ dyads only. Furthermore let

$$v_i^{\diamond}(z) = p(z)v_i^{\#}(z); \quad i = 1, 2, \ldots, m \tag{20}$$

then multiplying (18) by $p(z)$ and setting $z = z_0$ we get

$$[w_1^{\#}(z_0), w_2^{\#}(z_0), \ldots, w_m^{\#}(z_0)]\begin{bmatrix} 0_{m-\rho,m} \\ v_{m-\rho+1}^{\diamond \mathrm{T}}(z_0) \\ \cdot \\ \cdot \\ v_m^{\diamond \mathrm{T}}(z_0) \end{bmatrix} = 0_{m,m} \tag{21}$$

However, if $V^{\diamond}(z) = p(z)V^{\#}(z)$ we also have that

$$V^{\diamond}(z_0)W^{\#}(z_0) = 0_{m,m} \tag{22}$$

which implies that all the non-zero row vectors of $V^{\diamond}(z_0)$ must be parallel and thus must be in the same direction, say v_0

$$v_i^{\diamond}(z_0) = \alpha_i v_0; \quad i = m - \rho + 1, \ldots, m \tag{23}$$

The reason for this is that z_0 has geometric multiplicity 1, which implies that at z_0, the rank defect of $W^{\#}(z)$ is only 1, so that from (22) the rank of $V^{\diamond}(z_0)$ can only be 1. Substitution of (23) into (21) implies

$$W^{\#}(z_0)\begin{bmatrix} 0_{m-\rho,1} \\ \alpha_{m-\rho+1} \\ \cdot \\ \cdot \\ \alpha_m \end{bmatrix} = 0_{m,1} \tag{24}$$

Now, by its definition, the RCC of (15) can be written as

$$K(z) = \sum_{i=1}^{m} k_i(z)D_i(z) \tag{25}$$

which together with (18 *a*) implies that

$$K(z) - k_m(z)I = \sum_{i=1}^{m-1} s_{i,m}(z)D_i(z) \tag{26 a}$$

$$s_{i,j}(z) = k_i(z) - k_j(z) \tag{26 b}$$

From Lemma 4.1, z_0 is a candidate FM of $K(z)$, and will not be a FM if, and only if, it does not appear as a pole of $K(z)$, namely if $p(z_0)K(z_0) = 0$. By assumption none of the $k_i(z)$ have z_0 as a pole and hence the necessary and sufficient condition for z_0 not to be a FM is that the right-hand side of (26 *a*) vanishes at z_0. But this condition along with (23) imply

$$W^{\#}(z_0)\begin{bmatrix} 0_{m-\rho,1} \\ s_{m-\rho+1,m}(z_0)\alpha_{m-\rho+1} \\ \vdots \\ s_{m-1,m}(z_0)\alpha_{m-1} \\ 0 \end{bmatrix} = 0_{m,1} \tag{27}$$

However, as mentioned above, the rank defect of $W^{\#}(z_0)$ is 1, and thus $W^{\#}(z_0)$ has a one-dimensional kernel. Hence, comparing (27) with (24) we conclude that

$$s_{i,m}(z_0) = k_i(z_0) - k_m(z_0) = 0; \quad \text{for } i = m - \rho + 1, \ldots, m - 1 \tag{28}$$

which in turn implies that

$$s_{i,j}(z) = p(z)\phi_{i,j}(z); \quad \forall\, i,\, j = m - \rho + 1, \ldots, m \tag{29}$$

Restoring the $D_i(z)$ to their original order we obtain (19). □

The implication of the theorem above is clear: for an arbitrary choice of $k_i(z)$, all the simple poles of $V^{\#}(z)$ (i.e. all the poles of algebraic and geometric multiplicity 1) will be FMs of the RCC controller of (15). To avoid such FMs the $k_i(z)$ must be chosen so that condition (19) is satisfied. The theorem below states the corresponding condition for the case of poles of a general algebraic multiplicity.

Theorem 4.2: *Let z_0 be a pole of $V^{\#}(z)$ of algebraic multiplicity σ, and geometric multiplicity* 1, *and let $(z - z_0)^{\delta i}$ appear in the denominator of ρ_i dyads for $i = 1, 2, \ldots, \mu$ where clearly $\delta_\mu = \sigma$. Assume further that z_0 is not a pole of any of the eigenfunctions $k_i(z)$ of the RCC of* (15). *Then z_0 will not be a FM of $K(z)$, if and only if*

$$k_i(z) - k_j(z) = p^{\delta_{ij}}(z)\phi_{i,j}(z); \quad \delta_{i,j} = \min[\delta_i, \delta_j] \quad \forall\, i, j \in I \tag{30}$$

where $\phi_{ij}(z)$ is a rational function whose denominator is not a multiple of $p(z)$.

Proof: The proof of this result follows similar lines to those given for Theorem 4.1, hence here we shall not give all the detailed steps but rather we sketch an outline. Thus, reorder the multiplicities so that $\delta_i > \delta_j$ for $i > j$, and reorder the dyads $D_i(z)$ so that the last ρ_μ dyads are associated with δ_μ, the preceding $\rho_{\mu-1} - \rho_\mu$ are associated with $\delta_{\mu-1}$ etc, and define

$$v_i^{\diamond}(z) = p^{\delta_i}(z)v_i^{\#}(z); \quad i = m - \rho_1 + 1, m - \rho_1 + 2, \ldots, m \tag{31}$$

so that (22) still holds true for the matrix $V^{\diamond}(z)$ which compromises the above vectors as its row vectors. Since the geometric multiplicity of z_0 is assumed to be 1, the arguments in the proof of Theorem 4.1 can be used again to prove that (23) holds true when ρ is replaced by ρ_1. Next, observe that on account of (18) $\rho_\mu \geq 2$, multiply (18) and (26) by $p^\nu(z)$, $\nu = \delta_\mu$ and set $z = z_0$ to re-establish (24) and (27) so that

$$s_{i,j}(z) = p(z)s_{i,j}^{(1)}(z); \quad \forall\, i,\, j = m - \rho_\mu + 1, \ldots, m \tag{32}$$

where $s_{i,m}^{(1)}(z)$ is a rational function whose denominator is not a multiple of $p(z)$.

Equation (32) can be substituted into (26) so that a factor $p(z)$ is cancelled from the denominator of the last $\rho_\mu - 1$ dyads. Now (26) can be multiplied by

$(z - z_0)^\nu$, $\nu = \delta_\mu - 1$ and z can be set to z_0 so that in place of (32) we derive

$$s_{i,j}^{(1)}(z) = p(z) s_{i,j}^{(2)}(z); \quad \forall\, i,\, j = m - \rho_\mu + 1, \ldots, m \tag{33}$$

Substitution of (32) and (33) into (26) will have the effect of cancelling yet another factor $p(z)$ from the denominator of the last $\rho_\mu - 1$ dyads. This process can be repeated until the remaining power of $p(z)$ still present in the denominator of the last $\rho_\mu - 1$ dyads is equal to $\delta_{\mu-1}$. Thereafter, multiplication by $(z - z_0)^\nu$, $\nu = \delta_{\mu-1}$ will result in

$$\left.\begin{aligned} s_{ij}(z) &= [p(z)]^{\delta_\mu - \delta_{\mu-1}+1} s_{ij}^{(\delta_\mu - \delta_{\mu-1}+1)}(z);\ \forall\, i,\, j = m - \rho_\mu + 1, \ldots, m \\ s_{ij}(z) &= p(z) s_{ij}^{(1)}(z); \quad \forall\, i,\, j = m - \rho_\mu - \rho_{\mu-1} + 1, \ldots, m - \rho_\mu \end{aligned}\right\} \tag{34}$$

The whole procedure can be applied again to cancel further $p(z)$ terms from the last $\rho_\mu - \rho_{\mu-1}$ dyads until the highest remaining power is $\delta_{\mu-2}$ at which point the dyads whose multiplicity of $p(z)$ is $\delta_{\mu-2}$ will come into play. Carrying on in this manner we end up with the condition

$$\left.\begin{aligned} s_{ij}(z) &= [p(z)]^{\delta_\mu} s_{ij}^{(\delta_\mu)}(z); \quad \forall\, i,\, j = m - \rho_\mu + 1, \ldots, m \\ s_{ij}(z) &= [p(z)]^{\delta_\mu - 1} s_{ij}^{(\delta_\mu - 1)}(z); \quad \forall\, i,\, j = m - \rho_\mu - \rho_{\mu-1} + 1, \ldots, m - \rho_\mu \\ &\ \ \vdots \\ s_{ij}(z) &= p(z)^{\delta_1} s_{ij}^{(\delta_1)}; \quad \forall\, i,\, j = m - \rho_1 - \rho_2 + 1, \ldots, m - \rho_1 \end{aligned}\right\} \tag{35}$$

The proof can be completed by restoring the order of the dyads and associating the $s_{ij}(z)$ above with the $\phi_{ij}(z)$ of (30). □

The most general case of arbitrary geometric and algebraic multiplicities is governed by a similar result to that stated in Theorem 4.2.

Theorem 4.3: *Let z_0 be a pole of $V^{\#}(z)$ of algebraic multiplicity σ, and geometric multiplicity τ, and let $(z - z_0)^{\delta i}$ appear in the denominator of ρ_i dyads for $i = 1, 2, \ldots, \mu$ where clearly $\delta_\mu = \sigma$. Assume further that z_0 is not a pole of any of the eigenfunctions $k_i(z)$* of the RCC of *(15)*. Then z_0 *will not be a FM of $K(z)$ if, and only if*

$$k_i(z) - k_j(z) = p^{\delta_{ij}}(z) \phi_{i,j}(z); \quad \delta_{i,j} = \min[\delta_i, \delta_j] \quad \forall\, i,\, j \in I \tag{36}$$

where $\phi_{ij}(z)$ is a rational function whose denominator is not a multiple of $p(z)$.

Proof: The proof is identical to that given for Theorem 4.2 except that now the rank defect of $W^{\#}(z_0)$ is τ, which implies that $V^{\diamond}(z_0)$ has rank τ. As a consequence, the vector v_0 of (23) must be replaced by a $\tau \times m$ matrix V_0 and the scalars α_i will be $1 \times \tau$ vectors. From this point on the development of the proof is identical to that presented above. □

Remark 4.1: As will be seen in a later section the poles z_0 of $V^{\#}(z)$ bear a relationship to the BPs of $G(z)$ and this relationship suggests that, in all but very special cases, the geometric and algebraic multiplicity of z_0 will be 1. In the interest of keeping our presentation simple, in the following we shall assume this to be the case. □

The overall conclusion of § 4 is that unless the eigenfunctions $k_i(z)$ satisfy some constraints (such as the ones given in (19) for the case of simple poles) $K(z)$ will have FMs. In general, of course, $G(z)$ will not be the identity and so the FMs of $K(z)$ are of no direct interest. The section below explores the connections between FMs of $K(z)$ and FMs of $Q(z)$ and proves that unstable FMs of $K(z)$ are undesirable and can lead to instability.

5. Unstable FMs and stability

We begin with a result which appears to suggest that the FMs of $K(z)$ are of no consequence, and later explore this issue further.

Theorem 5.1: *Generically (the precise meaning of this will be made clear in the proof), FMs of* $K(z)$ *are not FMs of* $Q(z) = G(z)K(z)$.

Proof: Let z_0 be a simple (see Remark 4.1) pole of $V^{\#}(z)$ and without loss of generality assume that z_0 is not a pole or zero of $G(z)$. Then using the definition

$$Q^{\diamondsuit}(z) = p(z)Q(z); \quad p(z) = (z - z_0) \tag{37}$$

it is easy to show that $Q^{\diamondsuit}(z_0)$ will be unity rank; this follows from the fact that $V^{\diamondsuit}(z_0)$ is itself unity rank. Hence, if $p_Q(z)$ denotes the pole polynomial of $Q(z)$ we have

$$p_Q(z) = p(z)\tilde{p}_Q(z); \quad \tilde{p}_Q(z_0) \neq 0 \tag{38}$$

where $p_Q(z)$ is a polynomial, so that the closed-loop pole polynomial can be written as

$$\begin{aligned} p_R(z) &= p_Q(z)\det[I_m + kQ(z)] = p(z)\tilde{p}_Q(z)\prod_{i=1}^{m}(1 + kq_i(z)) \\ &= \frac{1}{p^{m-1}(z)}\tilde{p}_Q(z)\prod_{i=1}^{m}[p(z) + kq_i^{\diamondsuit}(z)] \end{aligned} \tag{39}$$

where use has been made of the fact that the eigenfunctions $q_i^{\diamondsuit}(z)$, $q_i(z)$ of $Q^{\diamondsuit}(z)$ and $Q(z)$ respectively satisfy the relationship $q_i^{\diamondsuit}(z) = p(z)q_i(z)$. Now, providing that the trace of $Q^{\diamondsuit}(z_0)$ is non-zero, all but one, say the jth, of the eigenvalues of $Q^{\diamondsuit}(z_0)$ will be zero, and hence we may write

$$\begin{aligned} q_i^{\diamondsuit}(z) &= p(z)q_i^{\square}(z); \quad \forall\, i \neq j \\ p_R(z) &= \tilde{p}_Q(z)[p(z) + kq_j^{\diamondsuit}(z)] \prod_{i=1, i\neq j}^{m} [1 + kq_i^{\square}(z)] \end{aligned} \tag{40}$$

Since $q_j^{\diamondsuit}(z)$ does not vanish at z_0, $p_R(z_0)$ cannot be identically zero for all values of k, and hence z_0 cannot be a FM of $Q(z)$.

In contrast to this, if the trace of $Q^{\diamondsuit}(z_0)$ is zero then all the eigenvalues of this matrix will be zero, $p_R(z_0)$ will be identically zero for all k, and z_0 will be a FM of $Q(z)$. However, generically the trace will not be zero. To see this, let $G(z)$ have a set of given denominator polynomials and arbitrary numerator polynomials. Then the trace of $Q^{\diamondsuit}(z_0)$ will be a linear function of the vector θ whose elements are the coefficients of the numerator polynomials. Thus, the trace can only be zero on a hyper-plane in the θ-space and will happen with probability zero. Furthermore, even if the plant transfer function matrix were such that θ lay on this hyper-plane, an arbitrary perturbation on the numerator coefficients of the numerators of $G(z)$ would move θ away from the hyper-plane with probability 1. □

Theorem 5.2: *If $K(z)$ commutes with $G(z)$, then FMs of $K(z)$ will also be fixed modes of $Q(z)$ unless the eigenfunctions $q_i(z)$ of $Q(z)$ satisfy* (19) *of Theorem 4.1.*

Proof: Without loss of generality we shall assume that z_0 is not a pole of the eigenfunctions $g_i(z)$, and $k_i(z)$ of $G(z)$ and $K(z)$, respectively. Nevertheless, providing that (19) is not satisfied by the $q_i(z)$, z_0 will be a pole of $Q(z)$ and so the analysis presented in the proof of Theorem 5.1 is valid. Now, under commutativity, we have

$$\text{trace}\,[Q^{\diamond}(z)] = p(z)\,\text{trace}\,[Q(z)] = p(z)\sum_{i=1}^{m} q_i(z) = p(z)\sum_{i=1}^{m} g_i(z)k_i(z) \quad (41)$$

so that this trace vanishes at z_0 and, by the proof of Theorem 5.1, z_0 will be a FM of $Q(z)$.

Finally if the $q_i(z)$ satisfy (19), then applying Theorem 4.1 to $Q(z)$ instead of $K(z)$ we conclude that z_0 cannot be a FM of $Q(z)$. □

Corollary 5.1: *If $G(z)$ does not have FMs, then all the FMs of a commutative controller will be FMs of $Q(z)$.*

Proof: Let z_0 be a FM of $K(z)$ so that by Theorem 4.1 we have that

$$k_i(z_0) - k_j(z_0) \neq 0 \quad \text{for some} \quad i, j \in I \quad (42)$$

Conversely, since by assumption $G(z)$ does not have FMs, the pole z_0 of its left eigenvector matrix cannot be a FM so that

$$g_i(z_0) - g_j(z_0) = 0 \quad \forall\, i, j \in I \quad (43)$$

Combining (42) and (43) we get that under commutativity

$$q_i(z_0) - q_j(z_0) = g_i(z_0)k_i(z_0) - g_j(z_0)k_j(z_0) \neq 0 \quad \text{for some } i, j \in I \quad (44)$$

which of course implies that z_0 is a FM of $Q(z)$. □

The stability implications of the corollary are clear: if $G(z)$ does not have FMs, and an exactly commutative controller has unstable FMs which are not FMs of $G(z)$, then the closed loop system of Fig. 1 will be unstable; this will be true for all possible choices of controller eigenfunctions $k_i(z)$ which do not remove the unstable FMs of $K(z)$.

Exact commutativity between $G(z)$ and $K(z)$ requires $G(z)$ to have rational eigenfunctions and this, in general, will not be the case; thus, the practical importance of Theorem 5.1 is limited. However, in the general case, exact commutativity can be replaced by near exact commutativity and then the FMs of $K(z)$ appear as 'near fixed modes' (NFM for short) of $Q(z)$; NFM are modes which, under the use of scalar gain k, remain close to their open-loop positions unless k assumes arbitrarily large values.

Lemma 5.1: *Let R_1 and R_2 be a small and a large real number and consider the annulus defined by the circles centred at the origin of radius R_1 and R_2. Puncture this annulus so as to remove all branch points and branch cuts of $G(z)$ and let the resulting domain be called D. Then for ν_i large enough, where ν_i denote the order of the eigenvector approximations used in Algorithm 2.1, $W^{\#}(z)$ can be made to give an arbitrarily good approximation to $W(z)$ everywhere in D.*

Proof: As ν_i tend towards $2\mu + 1$ so $W^{\#}(z)$ can be made to match the bi-causal expansion of $W(z)$ with arbitrarily small error. As a result, the frequency response of the error in the eigenvector approximation and hence the coefficients of its bi-causal expansion can also be made to be arbitrarily small. Since $W(z)$ and $W^{\#}(z)$ are analytic everywhere in D it follows (from arguments of analytic continuity) that $W^{\#}(z)$ can be made to give an arbitrarily good approximation to $W(z)$ anywhere in D. Clearly, the error of approximation will increase the further away we move from the unit circle, hence the need for the two circles of the lemma. □

Lemma 5.2: *The poles of $V^{\#}(z)$ that exist in the domain D of Lemma 5.1 lie in a small neighbourhood of a BP of $G(z)$ or lie on branch cuts of $G(z)$. Furthermore, every BP of $G(z)$ in D will have a pole of $V^{\#}(z)$ in its proximity.*

Proof: Since $W^{\#}(z)$ gives an arbitrarily good approximation of $W(z)$ everywhere in D and $W(z)$ becomes near singular only in the vicinity of a BP of $G(z)$, $W^{\#}(z)$ itself will be near singular if, and only if, z lies in the close proximity of BP of $G(z)$, or if z does not lie in D (i.e. if z lies outside the annulus of Lemma 5.1 or z lies on a branch cut). □

Theorem 5.3: *Under the conditions of Lemma 5.1 all the FMs of $K(z)$ that lie in D will be NFMs of $Q(z)$.*

Proof: Under the conditions of Lemma 5.1 we have that

$$\left.\begin{aligned} \Lambda[Q^{\diamond}(z)] &= \Lambda[W(z)\Lambda[G(z)]V(z)W^{\#}(z)\Lambda[K(z)]V^{\diamond}(z)] \\ &= \Lambda[V^{\diamond}(z)W(z)\Lambda[G(z)]V(z)W^{\#}(z)\Lambda[K(z)]] \\ &= p(z)\Lambda[G(z)]\Lambda[K(z)] + E(z) \end{aligned}\right\} \quad (45)$$

where $\Lambda(.)$ denotes the eigenvalue matrix of a transfer function, and $E(z)$ is arbitrarily small everywhere in D. The extreme right-hand side of (45) follows from the fact that $W^{\#}(z)$ is an arbitrarily good approximation of $W(z)$ so that under appropriate eigenvector scaling $V^{\#}(z)W(z)$ can be made to be arbitrarily close to I_m and $V^{\diamond}(z)W(z)$ arbitrarily close to $p(z)I_m$. Hence, from (39) we have that $p_R(z_0)$ is zero for $k = 0$, and arbitrarily small for all non-zero ks which are not arbitrarily large. Hence z_0 is a NFM of $Q(z)$. □

Thus, even though exact commutativity is not a practicable proposition, Theorem 5.2 shows that as $K(z)$ is made to commute with $G(z)$ to within a higher degree of accuracy, so the FMs of $K(z)$ will appear as poles of $Q(z)$ which are difficult to move through the use of scalar feedback, unless one is prepared to use arbitrarily large gains. There are a number of practical considerations which preclude the use of large gains for most realistic practical applications, hence once again the unstable FMs of $K(z)$ pose serious stability problems, for all possible choices of the eigenfunctions of $K(z)$ which do not remove these FMs.

Further evidence of possible stability difficulties caused by unstable FMs is provided by the theorem below.

Theorem 5.4: *Let r be the sum of all r_i, where r_i denotes the maximum number that the ith CL of $G(z)$ encircles (anticlockwise) the point $(-1/k + \mathrm{j}0)$ for all real values of k, and let n_p denote the number of unstable FMs of the RCC, $K(z)$. Furthermore assume that $G(z)$ and $K(z)$ are sufficiently commutative everywhere on the unit circle, in the sense that $V(z)W^{\#}(z)$ and $W^{\#}(z)V(z)$ are both approximately equal to I_m everywhere on the unit circle. Then, if $n_p > r - n_G^+$, where n_G^+ denotes the number of unstable poles of $G(z)$, the closed-loop system of Fig. 1 will either be unstable or will have very poor stability margins, for all possible choices of $k_i(z)$ which do not alter the value of n_p.*

Proof: Assume first that all the eigenfunctions, $k_i(z)$, of $K(z)$ are stable, let $G(z)$ have n_G^+ unstable poles, and let n_i' denote the number of anticlockwise encirclements of the critical point, $(-1 + \mathrm{j}0)$, by the CL of $Q(z)$. Then for stability we require that

$$\sum_{i=1}^{m} n_i' = n_G^+ + n_p > r \tag{46}$$

where the inequality above is a consequence of the assumption that $n_p > r - n_G^+$. Now since the $k_i(z)$ are all assumed to be stable, their frequency response will have a clockwise winding (Horowitz and Ben-Adam 1989) at all frequencies, namely $\mathrm{d}k_i(\mathrm{e}^{\mathrm{j}\theta})/\mathrm{d}\theta$ will point in a clockwise direction with respect to the centre of curvature of the frequency response of $k_i(z)$ at $z = \mathrm{e}^{\mathrm{j}\theta}$ for every θ. As a result, the net sum of critical point encirclements by the frequency response of $q_i^{\#}(z) = g_i(z)k_i(z)$ cannot be made to be greater than r_i. However, under the assumption of near commutativity between $G(\mathrm{e}^{\mathrm{j}\theta})$ and $K(\mathrm{e}^{\mathrm{j}\theta})$ the CL of $Q(z)$ will be approximately equal to the frequency response of $q_i^{\#}(z)$ and hence, in general, the left-hand side of (46) cannot be made to be greater than r. Since the two sets of frequency responses are not exactly equal, it is, of course, theoretically possible that the net sum of critical point encirclements given by $q_i^{\#}(z)$ will not be the same as that given by the CL, but the difference need not be large enough to satisfy (46) and, if it were, clearly the resulting stability margins would be extremely poor in that the use of a scalar feedback kI in the feedback loop of Fig. 1 would result in instability for values of k which were close to but not equal to 1.

Finally, unstable $k_i(z)$ can, of course, increase the left-hand side of (46) but at the same time they will also increase the right-hand side (by the same amount) because the Nyquist criterion requires that the net sum of anticlockwise critical point encirclements by the CL of $Q(z)$ be equal to the total number of unstable poles of $G(z)$ and $K(z)$. Hence, the closed-loop system of Fig. 1 will still remain unstable, or would have poor stability margins. □

It is pointed out that the value of r can be very small, typically for most stable $G(z)$ r would be zero. On the other hand, depending on the order of eigenvector approximations used to derive $W^{\#}(z)$, n_p can be large; typically, by Lemma 5.2, it is not unreasonable to expect n_p to be comparable to the number of unstable BP of $G(z)$. Therefore, the inequality $n_p > n_G^+ - r$ of Theorem 5.3 is not a strong condition and will be met by a large class of $G(z)$; in all such cases the presence of unstable FMs in $K(z)$ would pose serious stability problems.

6. Design study

The results of § 5 suggest that practical RCCs must not have unstable FMs and § 4 provides the mechanism through which such FMs can be avoided. In particular, Theorem 4.2 indicates that the eigenfunctions of the RCC cannot be chosen arbitrarily but they must satisfy condition (19); $p(z)$ is the product of terms $(z - z_i^+)$, where z_i^+ denotes the unstable (and/or undesirable) poles of $V^{\#}(z)$. Any undesirable poles of $V^{\#}(z)$ which are not roots of $p(z)$ will become FMs of $K(z)$ and thus may lead into the stability (and/or relative stability) difficulties discussed in § 5.

Thus, the problem of designing RCCs comprises two tasks: (i) the computation of suitable eigenvector approximations; (ii) the choice of eigenfunctions $k_i(z)$ which have the desired effect on the gain/phase characteristics of the CL of $G(z)$ while at the same time satisfy condition (19). The former task is straightforward and can be achieved through the use of Algorithm 2.1. Task (ii) however is more complex in that the choice of $k_i(z)$ which improves the gain/phase characteristics of the CL of $G(z)$ is by no means unique. What is needed is a characterization of the family of $k_i(z)$ which is acceptable from a practical viewpoint and which provides adequate CL compensation. Then one can give up the degrees of freedom available in this characterization in order to ensure that condition (19) is satisfied. The systematic treatment of this topic falls beyond the scope of the present paper; here we give a numerical example that illustrates the results of § 4.

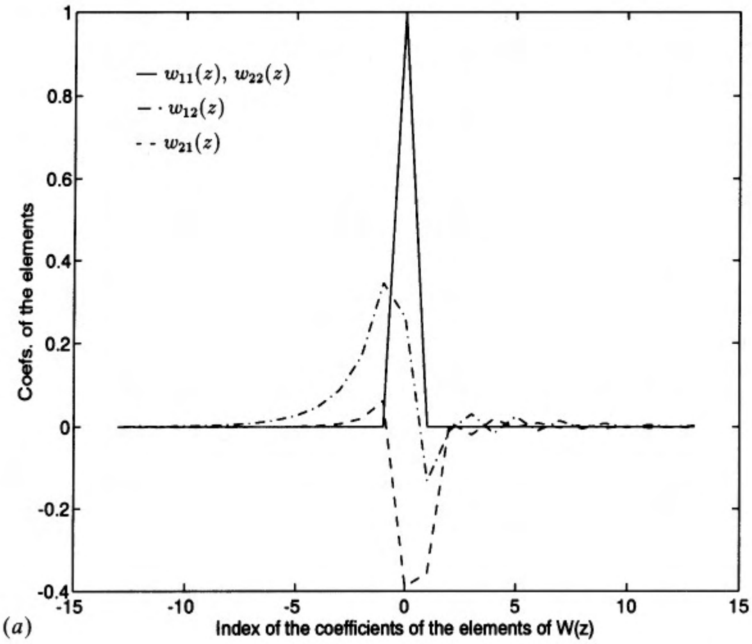
Thus, consider the transfer function matrix (Cloud and Kouvaritakis 1987) whose elements $g_{ij}(z)$ are given as

$$g_{11}(z) = \frac{0{\cdot}7682z^{-1} + 0{\cdot}2372z^{-2} - 0{\cdot}2702z^{-3} - 0{\cdot}0622z^{-4} - 0{\cdot}0007z^{-5}}{1 - 1{\cdot}33z^{-1} + 0{\cdot}11z^{-2} + 0{\cdot}29z^{-3} - 0{\cdot}03z^{-4}} \tag{47 a}$$

$$g_{12}(z) = \frac{0{\cdot}1113z^{-1} + 0{\cdot}2834z^{-2} + 0{\cdot}1493z^{-3} - 0{\cdot}0252z^{-4} - 0{\cdot}0056z^{-5}}{1 - 1{\cdot}3241z^{-1} + 0{\cdot}037z^{-2} + 0.3494z^{-3} - 0{\cdot}0354z^{-4}} \tag{47 b}$$

$$g_{21}(z) = \frac{-0{\cdot}1256z^{-1} - 0{\cdot}053z^{-2} + 0{\cdot}2866z^{-3} + 0{\cdot}2666z^{-4} + 0{\cdot}0374z^{-5} - 0{\cdot}0149z^{-6}}{1 - 1{\cdot}135z^{-1} - 0{\cdot}4613z^{-2} + 0{\cdot}7449z^{-3} - 0{\cdot}0445z^{-4} - 0{\cdot}0909z^{-5} + 0{\cdot}0145z^{-6}} \tag{47 c}$$

$$g_{22}(z) = \frac{0{\cdot}4992z^{-1} + 0{\cdot}4147z^{-2} - 0{\cdot}0826z^{-3} - 0{\cdot}0564z^{-4} + 0{\cdot}0449z^{-5} + 0{\cdot}0132z^{-6}}{1 - 1{\cdot}162z^{-1} - 0{\cdot}4434z^{-2} + 0{\cdot}7687z^{-3} - 0{\cdot}052z^{-4} - 0{\cdot}0941z^{-5} + 0{\cdot}0152z^{-6}} \tag{47 d}$$

(*a*)


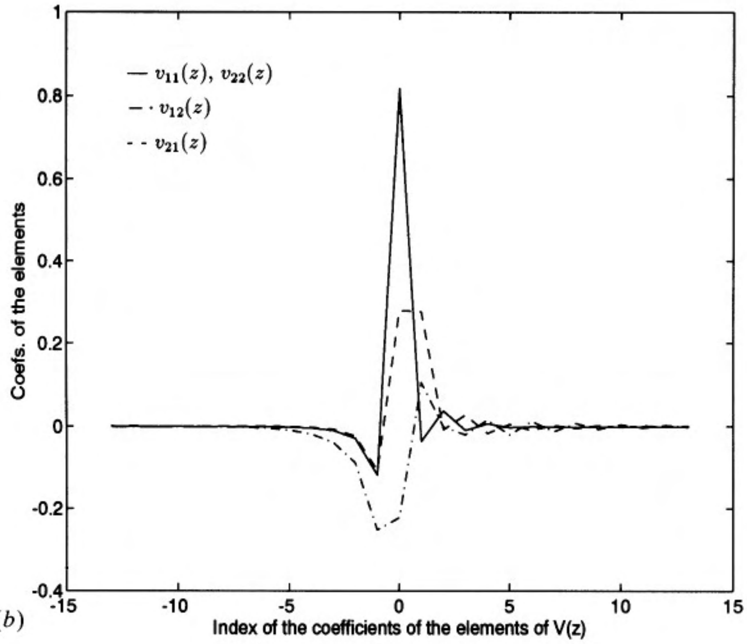
(*b*)


Figure 4. Bi-causal expansion of the elements of $W(z)$ – $\mu = 13$ (*a*); Bi-causal expansion of the elements of $V(z)$ – $\mu = 13$ (*b*).

The bicausal eigenvector sequences for this $G(z)$ are shown in Figs 4(a), (b) from which it can be seen that μ can be taken to be 13. With this value of μ and for $N = 27$, the algorithm of § 2 gives, the following third-order approximation to $W(z)$

$$W^{\#}(z) = \begin{bmatrix} -0{\cdot}8533 & -0{\cdot}5516 \\ 0{\cdot}4726 & 1{\cdot}3611 \end{bmatrix} + \begin{bmatrix} 1{\cdot}8868 & -1{\cdot}3930 \\ -0{\cdot}3599 & -1{\cdot}4468 \end{bmatrix} z^{-1} + \begin{bmatrix} 1{\cdot}1890 & -0{\cdot}5207 \\ -1{\cdot}1526 & -2{\cdot}2809 \end{bmatrix} z^{-2} + \begin{bmatrix} -0{\cdot}1132 & 0{\cdot}2759 \\ -0{\cdot}3440 & -0{\cdot}3021 \end{bmatrix} z^{-3} \quad (48)$$

and the misalignment angles between the first and second columns of $W^{\#}(z)$ and the corresponding columns of $W(z)$ computed at the preselected frequencies, are small as shown in Fig. 5. For this choice of $W^{\#}(z)$, $V^{\#}(z)$ has two unstable poles located at 4·1279 and 2.0458 which must be included as roots of the $p(z)$ of condition (19) if $K(z)$ is not to have these unstable poles as FMs; the corresponding $p(z)$ is given by

$$p(z) = z^{-2}(z - 4{\cdot}1296)(z - 2{\cdot}0458) = 1 - 6{\cdot}1754z^{-1} + 8{\cdot}4484z^{2} \quad (49)$$

The CL of $G(z)$ are shown in Fig. 6, from which it is apparent that some phase advance would be advantageous to both CL, although of course more phase advance is required for $g_2(z)$. With this in mind Cloud and Kouvaritakis (1987), choose

$$k_1(z) = 0{\cdot}81, \quad k_2(z) = \frac{1{\cdot}36(z - 0{\cdot}63)}{z + 0{\cdot}0855} \quad (50)$$

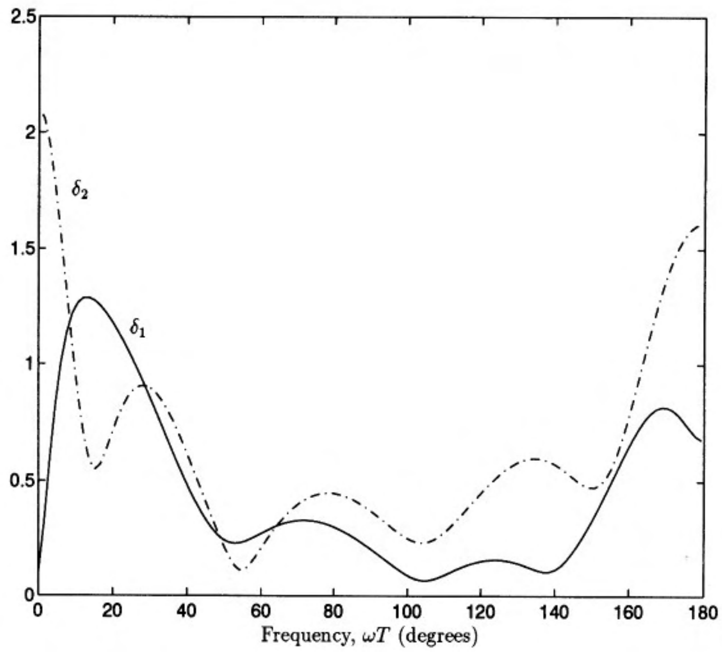


Figure 5. Misalignment angles between $W(z)$ and $W^{\#}(z)$.

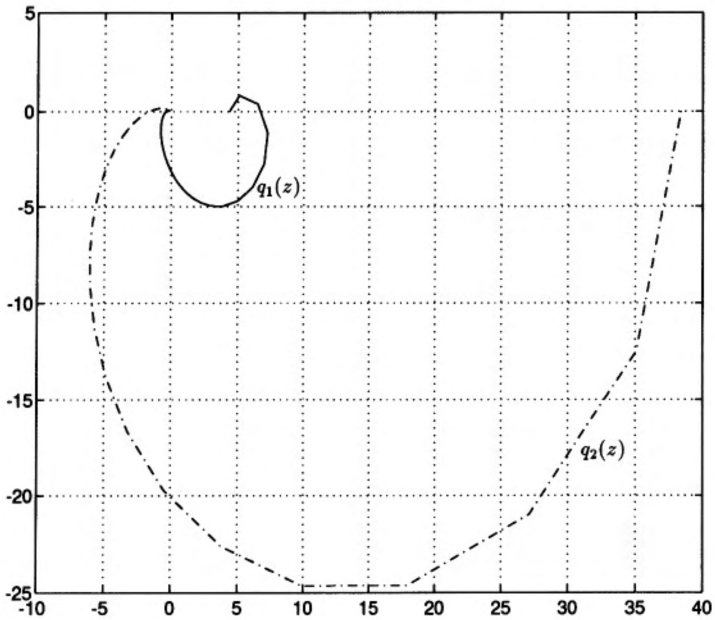


Figure 6. Characteristic loci of $G(z)$.

this choice however does not meet condition (19). To overcome this problem, here we replace $k_1(z)$ and $k_2(z)$ by

$$k_1(z) = \frac{0{\cdot}985(z - 0{\cdot}9544)(z - 0{\cdot}4237)}{(z - 0.9)(z + 0.085)}; \quad k_2(z) = \frac{(z - 0{\cdot}75)(z - 0{\cdot}7)}{(z - 0{\cdot}9)(z + 0{\cdot}085)} \tag{51}$$

where, for convenience, we choose $k_1(z)$ and $k_2(z)$ to share a common denominator. As required, the frequency response of these eigenfunctions, shown in Fig. 7(a), provides a phase advance at high frequencies with the larger amount of phase advance being present in $k_2(z)$. The corresponding target CL of $Q(z) = G(z)K(z)$ are shown in Fig. 7(b) and can be seen to have adequate gain/phase margins.

Finally, Fig. 8 gives the percentage error between these target CL and the ones actually achieved by the RCC; clearly, on account of the high degree of commutativity indicated by the low misalignment angles of Fig. 5, the accuracy of the eigenvalue compensation achieved by the RCC is very high and is several orders of magnitude better than that of the controller used by Cloud and Kouvaritakis (1987). Indeed the percentage error of our RCC is less than 0·05% for both CL and over all frequencies, whereas the corresponding error for the commutative controller of the earlier work is more than 8% and gets close to 20% for $q_2(z)$. The RCC presented here also outperforms (again by several orders of magnitude) the CCC of Kouvaritakis and Basilio (1994), although of course this is due to the fact that Theorem 4.1 enabled us to define 'achievable' targets.

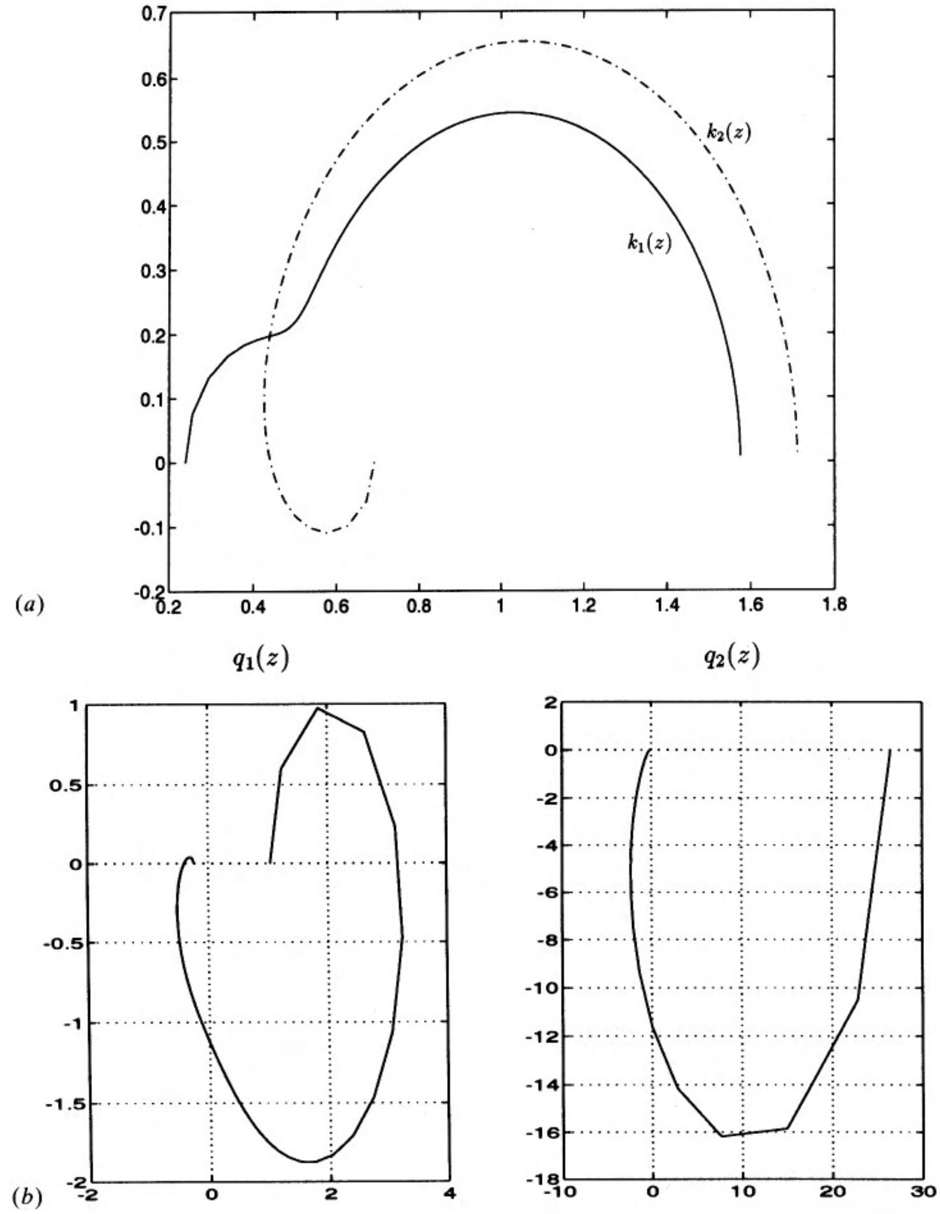


Figure 7. Frequency responses of $k_1(z)$ and $k_2(z)$ (a); desired characteristic loci of $Q(z)$ (b).

7. Conclusions

This paper explored difficulties associated with the presence of FMs in commutative controllers, which are based on rational eigenvector approximations. Furthermore, necessary and sufficient conditions for avoiding unstable/undesirable FMs were developed; these conditions effectively define a class of achievable targets for the gain/phase compensation of CL. An earlier paper

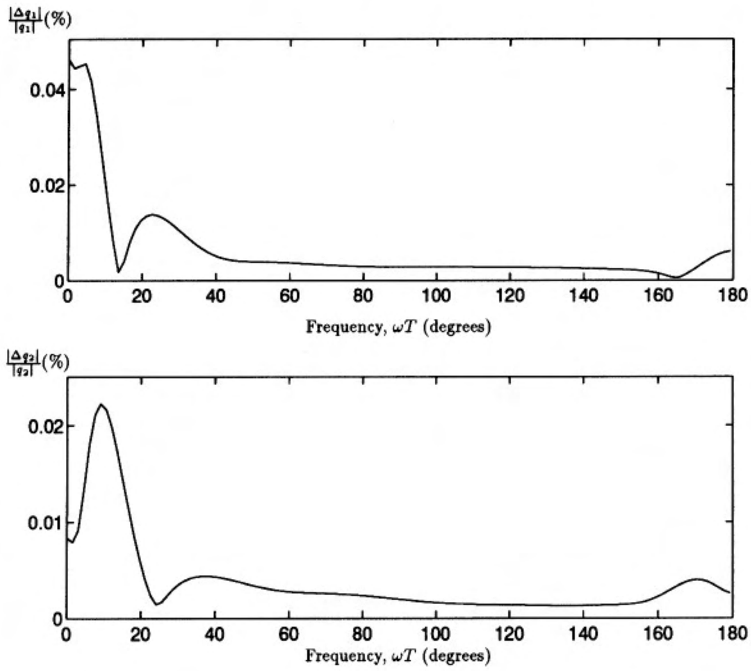


Figure 8. Percentage error between desired and achieved CL of $Q(z)$.

showed that causal commutative controllers can be constructed using bi-causal sequences for both the controller eigenvectors and eigenvalues. In terms of RCCs this would seem to imply that one need not restrict attention to stable controller eigenfunctions. The derivation of the whole class of achievable targets as well as the systematic use of the necessary and sufficient conditions of § 4 for the purposes of design form the topic of future research.

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