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Bi-causal eigenvector sequences and the design of causal commutative controllers

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The use of Laurent power series expansions of the eigenvector matrix of a linear multivariable transfer function matrix $G(z)$ holds the key to the physical realization of commutative controllers. In general, however, such controllers would be anti-causal. It is the purpose of this paper to show that there are enough degrees of freedom in the choice of the controller eigenfunctions both to effect gain/phase compensation of the frequency response of the eigenfunctions of $G(z)$, and to force the resulting control law to be causal. The results of the paper are shown to be superior to those possible through the use of approximate commutative controllers proposed earlier.

1. Introduction

In accordance with the generalized Nyquist criterion (MacFarlane and Postlethwaite 1977), a multivariable discrete-time system with open-loop transfer function matrix $G(z)$ will be stable under unity feedback if and only if the generalized Nyquist diagram of $G(z)$ gives the prerequisite number of critical point encirclements. The generalized Nyquist diagram comprises a set of plots, the 'characteristic loci', defined as the frequency response plots of the eigenfunctions of $G(z)$, $g_i(z)$, $i = 1, 2, \ldots, m$; m denotes the number of system inputs (assumed to be equal to the number of outputs). Over and above their use in assessing absolute stability margins, the characteristic loci can also be used to assess relative stability margins. The arguments underpinning this property are based on analyticity/conformality and are exactly the same as the arguments used in the case of scalar systems. It is not surprising therefore that, in this context, the ability to adjust the gain/phase characteristics of the characteristic loci has a key role to play in design.

From the theory of algebraic functions of a complex variable it is known that $g_i(z)$ for $i = 1, 2, \ldots, m$, viewed as the branches of the characteristic gain function $g(z)$ which satisfies the equation $\det[g(z)I - G(z)] = 0$, are analytic and locally distinct everywhere except at branch points. Thus, everywhere except at branch points, $G(z)$ will have a complete set of m linearly independent right eigenvectors $\underline{w}_i(z)$ and dual left eigenvectors $\underline{v}_i^{\mathrm{t}}(z)$, for $i = 1, 2, \ldots, m$ so that

$$G(z) = \sum_{i=1}^{m} g_i(z)\underline{w}_i(z)\underline{v}_i^{\mathrm{t}}(z) = W(z)\Lambda_G(z)V(z) \tag{1}$$

where $W(z)$ denotes the eigenvector matrix of $G(z)$ comprising $\underline{w}_i(z)$ as its column vectors, and $V(z) = W^{-1}(z)$ is the matrix comprising $\underline{v}_i^{\mathrm{t}}(z)$ as its row vectors; implicit in the above is the assumption that the eigenvectors have been

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scaled appropriately so that $\underline{v}_i^t(z)\underline{w}_i(z) = 1$. Equation (1) suggests an elegant approach (MacFarlane and Belletrutti 1973) to the design problem, namely the determination of a pre-compensator $K(z)$ that enables the systematic adjustment of eigenfunctions:

$$K(z) = \sum_{i=1}^{m} k_i(z)\underline{w}_i(z)\underline{v}_i^t(z) = W(z)\Lambda_K(z)V(z) \tag{2}$$

where $k_i(z)$, $i = 1, 2, \ldots, m$, denote the eigenfunctions of $K(z)$. It is easy to show that the eigenfunctions $q_i(z)$ of the compensated open-loop transfer function matrix $Q(z) = G(z)K(z)$ are those of $G(z)$ multiplied by the corresponding eigenfunctions of $K(z)$, namely that $q_i(z) = g_i(z)k_i(z)$; it is assumed that $g_i(z)$ and $k_i(z)$ share the eigenvector $\underline{w}_i(z)$ in common. This controller is called a commutative controller because it commutes under multiplication with $G(z)$. Despite its theoretical convenience this approach presents some practical difficulties: (i) for $m > 2$, in all but very special cases $W(z)$ and $V(z)$ will not be known explicitly; (ii) in general $W(z)$ and $V(z)$ are irrational functions of z.

To circumvent these difficulties, earlier work (MacFarlane and Kouvaritakis 1977) proposed the use of approximate commutative controllers (ACC), which have the structure shown in (2) but with $W(z)$ and $V(z)$ replaced by W_a and V_a, respectively; W_a and V_a are two matrices of real constants chosen to approximate, in a suitable sense, $W(z)$ and $V(z)$ at some prescribed frequency ω_0. This scheme yields controllers that are easy to implement; however, commutativity is now only approximate and is limited to a range of frequencies about ω_0. As a result $q_i(z)$ is only approximately equal to $g_i(z)k_i(z)$ for frequencies close to ω_0.

Subsequent work (Cloud and Kouvaritakis 1987) tackled this problem by introducing causal power series approximations $W_a(z)$ and $V_a(z)$ in place of W_a and V_a. It can be shown that, if all branch points of the characteristic gain function $g(z)$ are stable, then the sequences of the matrix coefficients of such power series are convergent, and hence the resulting approximate commutative controller can be made to commute with $G(z)$ (on the unit circle) to within any desired degree of accuracy simply by including higher powers of z^{-1} in $W_a(z)$ and $V_a(z)$; for this reason, the controllers derived in this manner are referred to as approximately exact commutative controllers (AECC). In the case of unstable branch points, however, the sequences of coefficients diverge and truncation becomes necessary. The theory of asymptotic expansions (Murray 1974, Kouvaritakis *et al.* 1990) can be invoked in order to determine the optimal level of truncation, and to stipulate upper bounds on the error of approximation. The commutative controllers derived in this way, though no longer approximately exact, are much superior to the approximately commutative controller; they give a higher degree of commutativity over a wider range of frequencies.

To obtain (nearly) exact commutativity in the case of unstable branch points, it is necessary to use bi-causal power series representations $W_{bc}(z)$ and $V_{bc}(z)$ for $W(z)$ and $V(z)$. It can be shown (Kouvaritakis and Rossiter 1991) that so long as no branch point of $G(z)$ lies on the unit circle and that no branch cut crosses the unit circle, the sequence of matrix coefficients, both for the causal and anti-causal part of the eigenvector representation, are convergent; therefore, the controller indicated by (2) with $W_{bc}(z)$ and $V_{bc}(z)$ in place of W_a and V_a can be made to commute with $G(z)$ (on the unit circle) to within any desired degree of accuracy. This type of controller will be referred to as a nearly exactly commutative controller (NECC). In general, such a controller would be anti-causal and could not be implemented in real life. It is the purpose of the present paper to show that this need not be so, even if $\Lambda_K(z)$ itself is chosen to be an anti-causal operator. This counter-intuitive innovation introduces extra degrees of freedom which can be given up in order to ensure that the overall controller both (i) is causal; and (ii) can be used for the systematic gain/phase adjustment of the characteristic loci of $G(z)$ over all frequencies.

A brief review of the mathematical background to the power series representations of eigenfunctions and eigenvector functions is given in § 2, and necessary and sufficient conditions for the causality of commutative controllers are derived; this is done for the 2×2 case first, and is subsequently extended to the general $m \times m$ case. The implied characterization of the degrees of freedom of commutative controllers is exploited in § 3, where an algorithm is given for the systematic adjustment of the gain/phase characteristics of generalized Nyquist diagrams. The superiority of the results of the paper over those possible through the use of other forms of commutative controllers is illustrated, by means of a design study, in § 4. Finally, the conclusions of the paper are drawn in § 5.

2. The conditions for commutativity and causality

2.1. *Mathematical preliminaries*

Multiplication of a matrix by a scalar does not affect the eigenvectors of the matrix. It is convenient here to multiply $G(z)$ by the least common denominator of all the elements of $G(z)$ in order to get $N(z) = d(z)G(z)$ and examine the eigen-properties of the numerator polynomial matrix $N(z)$ instead of $G(z)$. The main advantage of this is that the only singular points of $n(z)$, the characteristic gain function of $N(z)$, are branch points; the same observation therefore applies to the eigenvector functions of $N(z)$ (which of course are also eigenvector functions of $G(z)$) as well. Then by an appropriate application of Laurent's theorem and a suitable form of the inverse sampling theorem we may state (Kouvaritakis and Rossiter 1991) the following theorem.

Theorem 2.1: *If the unit circle (centred at the origin of the z-plane) does not go through any branch points of $n(z)$ and does not cross any branch cuts, then each branch $n_i(z)$, $i = 1, 2, \ldots, m$, has a distinct Laurent expansion for which the sequence of coefficients $\{n_0, n_1, n_2, \ldots\}$ of powers of z^{-1} and the sequence of coefficients $\{n_{-1}, n_{-2}, \ldots\}$ of powers of z, both taken in ascending order, will converge to zero. Furthermore if $\hat{n}^{(i)}$ denotes the vector of the sampled frequency response of $n_i(z)$, whose* kth *element is $n_i(\exp(\mathrm{j}k2\pi/(2\mu+1)))$ for $k = 0, 1, 2, \ldots, 2\mu$, then to within aliasing errors the causal and anti-causal sequences of coefficients are defined by*

$$n^{(i)} = [n_0, n_1, \ldots, n_\mu, n_{-\mu}, n_{-\mu+1}, \ldots, n_{-1}]^{\mathrm{T}} = \frac{1}{2\mu+1} F^* \hat{n}^{(i)} \tag{3}$$

where F is the $(2\mu+1) \times (2\mu+1)$ matrix whose $p+1$, $q+1$ element is $\exp(-\mathrm{j}pq2\pi/(2\mu+1))$ *and where $(\cdot)^*$ denotes transposition and complex conjugation. The aliasing error in the inverse sampling process described in* (3) *tends to zero as μ becomes arbitrarily large.*

Proof: See Kouvaritakis and Rossiter (1991). □

Over and above the convergence of the causal and anti-causal sequences of coefficients in the Laurent expansion of $n^{(i)}(z)$, the theorem above prescribes an efficient means for the computation of these sequences: sample $n^{(i)}(z)$ at $2\mu + 1$ equispaced points around the unit circle, introduce the vector of sampled values into (3) to get a vector whose elements initially assume some finite (non-zero) values, then tend to zero and then rise to some finite non-zero values. For convenience, implicit in the theorem above is the assumption that the number of significant terms in the strictly causal (i.e. $n_1, n_2, \ldots, n_\mu$) and anti-causal sequences are the same. In practice this will differ depending on the proximity of the stable and unstable branch points to the unit circle. So, for example, if the unstable branch points are much further away from the unit circle than the stable ones, then the anti-causal sequence will converge to zero much faster than the causal sequence. As a result the tail of the anti-causal sequence, say the last v terms, will be negligibly small. In such an instance, therefore, one need only sample at $2\mu + 1 - v$ points around the unit circle and then use (3) to obtain the causal sequence by considering the first $\mu + 1$ elements of $n^{(i)}$, and use the remaining elements to form the anti-causal sequence. For clarity of exposition, and without loss of generality, in what follows all causal sequences will be taken to be $\mu + 1$ long and all strictly anti-causal sequences will be taken to be μ long.

The eigenvectors of $N(z)$ share the same branch points and branch cuts with the eigenfunctions $n_i(z)$ (and indeed are defined on the same Riemman surface). Therefore, the convergence properties of Theorem 1 can also be asserted with respect to $W(z)$ and $V(z)$, whereas the inverse sampling procedure for computing the causal and anti-causal sequences of coefficients could be applied to the individual elements of $W(z)$ and $V(z)$. As a result, the eigenvector matrices of $N(z)$ can be represented in terms of the bi-causal power series.

$$W_{bc}(z) = \sum_{k=-\mu}^{\mu} W_k z^{-k} \quad \text{and} \quad V_{bc}(z) = \sum_{k=-\mu}^{\mu} V_k z^{-k} \tag{4}$$

Unlike eigenfunctions, which are unique, eigenvector functions are subject to an arbitrary scaling factor. It is possible to choose this with the view to minimizing and balancing the length of the anti-causal component in $\underline{w}_i(z)$ and the corresponding $\underline{v}_i^T(z)$; algorithms for achieving this can be found in Kouvaritakis *et al.* (1990), and Rossiter and Kouvaritakis (1991). For our purposes here, the choice of scaling factors is not of particular interest; we will however assume that, whatever the eigenvector scaling used is, it results in continuous and smooth frequency response plots for the elements of $W(z)$ (and hence those of $V(z)$). Under such circumstances, μ can be chosen to be large enough so that the controller

$$K(z) = W_{bc}(z)\Lambda_K(z)V_{bc}(z) \tag{5}$$

can be made to commute, to within any degree of accuracy, with $G(z)$ anywhere on the unit circle; this, after all, is the justification for the term nearly exact commutative controller (NECC).

The problem with (5), however, is that in general, for an arbitrary choice of $\Lambda_K(z)$, the implied NECC would be anti-causal. Yet if $\Lambda_K(z)$ where chosen to be, say $[\Lambda_G(z)]^i$, with i any positive integer, then the NECC would be (to within truncation errors) equal to $G^i(z)$ and hence would be causal. Indeed, taking the eigenfunctions of $K(z)$ to be $f(g_i(z))$, where $f(\cdot)$ is any function that is analytic inside the unit disc, would result in a causal NECC. Thus, there exists a whole family of $\Lambda_K(z)$ for which $K(z)$ is causal and for the purposes of design, therefore, what is needed is a characterization of the available degrees of freedom in the choice of the controller eigenfunctions. We begin by considering the 2×2 case first. This is done for two reasons: (*a*) clarity of exposition; (*b*) in the 2×2 case it is possible to state the conditions for causality directly in terms of the individual elements of $N(z)$.

2.2. *The characterization of causal exact commutative controllers for the 2 × 2 case*

Let n_{ij} for i, $j = 1$, 2 denote the elements of $N(z)$, and let D denote the discriminant of the characteristic equation of $N(z)$. Then it is easy to show that the eigenvector and dual eigenvector matrices of $N(z)$ can be written as

$$W(z) = \begin{bmatrix} n_{11} - n_{22} + \sqrt{D} & n_{11} - n_{22} - \sqrt{D} \\ 2n_{21} & 2n_{21} \end{bmatrix};$$

$$V(z) = \frac{1}{4n_{21}\sqrt{D}} \begin{bmatrix} 2n_{21} & -n_{11} + n_{22} + \sqrt{D} \\ -2n_{21} & n_{11} - n_{22} + \sqrt{D} \end{bmatrix} \tag{6}$$

so that with $k_1(z)$ and $k_2(z)$ as eigenfunctions, the corresponding commutative controller could be written as

$$\begin{aligned} K(z) = {} & \frac{k_1(z) + k_2(z)}{2} I \\ & + \frac{k_1(z) - k_2(z)}{2\sqrt{D}} \begin{bmatrix} n_{11}(z) - n_{22}(z) & 2n_{12}(z) \\ 2n_{21}(z) & -(n_{11}(z) - n_{22}(z)) \end{bmatrix} \end{aligned} \tag{7}$$

Thus we may state the following result.

Lemma 2.1: *A 2×2 commutative controller with eigenvalues $k_1(z)$ and $k_2(z)$ will be causal if and only if $[k_1(z) + k_2(z)]$ and $[k_1(z) - k_2(z)]/\sqrt{D}$ are both analytic outside the unit disc.*

Proof: The requirement for causality is equivalent to insisting that $K(z)$ should have a Maclaurin expansion in negative powers of z, namely that $K(z)$ be analytic outside the unit disc. Therefore, consideration of the off-diagonal elements implies that $[k_1(z) - k_2(z)]/\sqrt{D}$ must be analytic outside the unit disc and, subsequently, consideration of the diagonal elements implies that $[k_1(z) + k_2(z)]$ must also be analytic outside the unit disc. The sufficiency of these conditions for the causality of $K(z)$ is obvious. □

In the special case when the characteristic function $n(z)$ of $N(z)$ has only stable branch points, the conditions of Lemma 2.1 would be satisfied for any $k_1(z)$ and $k_2(z)$ which are analytic outside the unit disc. In such a case, as explained earlier, the commutative controller of (7) can be implemented as an approximately exact commutative controller using the power series eigenvector approximations $W_a(z)$ and $V_a(z)$. In the general case, however, there will be a number of unstable branch points, say $2\gamma_+$; this number will be even because of the assumption made that no branch cuts of $n(z)$ should cross the unit circle. Thus, for this case the discriminant D will factorize as $D = [D_-(z)][z^{-(2\gamma+)}D_+(z)]$, with the zeros of $D_-(z)$ all lying inside the unit disc and all the $2\gamma_+$ roots of $D_+(z)$ lying outside the unit circle. Accordingly, the $1/\sqrt{D}$ term in (7) will be given by the product $[1/\sqrt{D_-(z)}][z^{\gamma+}/\sqrt{D_+(z)}]$ whose first and second factors admit, respectively, a causal and anti-causal power series expansion. Furthermore, the term $1/\sqrt{D_+(z)}$ itself admits an anti-causal expansion and hence we are led to the following result.

Lemma 2.2: *A 2×2 commutative controller will be causal if, and only if, its eigenfunctions $k_1(z)$ and $k_2(z)$ can be written as*

$$k_1(z) = f_1(z) + f_2(z)z^{-\gamma_+}\sqrt{D_+(z)}; \quad k_2 = f_1(z) - f_2(z)z^{-\gamma_+}\sqrt{D_+(z)} \tag{8}$$

where $f_1(z)$ and $f_2(z)$ are any two function which are analytic outside the unit disc.

Proof: The sufficiency of the result is obvious from (7), whereas the necessity follows from the fact that $k_1(z) - k_2(z)$ must have $\sqrt{D_+(z)}/(z^{\gamma+})$ as a factor and the fact that $k_1(z) + k_2(z)$ must be analytic outside the unit disc. □

For the purposes of implementation, rather than use (8) it is possible to use power series representations in terms of negative powers of z for $f_1(z)$ and $f_2(z)$ and positive powers of z for $\sqrt{D_+(z)}$:

$$\left.\begin{aligned} &f_1(z) = a_0 + a_1z^{-1} + \cdots + a_\alpha z^{-\alpha}; \quad f_2(z) = b_0 + b_1z^{-1} + \cdots + b_\beta z^{-\beta} \\ &z^{-\gamma_+}\sqrt{D_+(z)} \simeq d_{-\mu}z^{\mu_+}d_{-\mu+1}z^{\mu-1} + \cdots + d_{-1}z + d_0 + d_1z^{-1} + \cdots + d_{\gamma_+}z^{-\gamma_+} \end{aligned}\right\} \tag{9 a, b, c}$$

and then perform the convolutions implied by (8); as explained earlier, μ can be chosen to be large enough so that (9 c) holds with an equality sign to within any desired degree of accuracy, whereas the positive integers α and β can be chosen to be large enough so that $f_1(z)$ and $f_2(z)$ represent any general functions analytic outside the unit disc (to within any desired accuracy). Note that the coefficients d_{-i}, $i = 1, 2, \ldots, \mu$, can be obtained by an inverse discrete Fourier transformation, similar to that used in Theorem 2.1. These observations combine to give the following result.

Theorem 2.2: *Consider Laurent expansions of the eigenfunctions $k_1(z)$ and $k_2(z)$ which are truncated after the z^μ and $z^{-\mu}$ term, and let $\underline{k}_1$ and $\underline{k}_2$ be the vectors formed out of the coefficients (taken in descending order of powers of z) of the Laurent expansions of $k_1(z)$ and $k_2(z)$. Then, to within truncation errors, in the 2×2 case the NECC defined by eqns. 4 and 5 will be causal if, and only if*

$$\left.\begin{aligned} &\begin{bmatrix} \underline{k}_1 \\ \underline{k}_2 \end{bmatrix} = \begin{bmatrix} E & C_{D_+} \\ E & -C_{D_+} \end{bmatrix} \underline{\theta} \\ &\text{where} \quad E = \begin{bmatrix} 0_{\mu,\mu+1} \\ I_{\mu+1,\mu+1} \end{bmatrix} \quad \text{and} \quad \underline{\theta} = \begin{bmatrix} \underline{f}_1 \\ \underline{f}_2 \end{bmatrix} \end{aligned}\right\} \tag{10}$$

The vectors $\underline{f}_1$ and $\underline{f}_2$ comprise the coefficients (taken in descending order) of the expansions for $f_1(z)$ and $f_2(z)$ given in (9 a), (9 b); without loss of generality it has been assumed that $\alpha = \mu$, and $\beta = \mu - \gamma_+$. The matrix C_{D_+} is a lower triangular striped Toeplitz matrix of dimension $(2\mu + 1) \times (\mu + 1)$ formed out of the coefficients d_i, $i = -\mu, -\mu + 1, \ldots 0, 1, \ldots, \gamma_+$ of the Maclaurin expansion for $z^{-(\gamma+)}\sqrt{D_+(z)}$ of (9 c).

Proof: This result is a restatement of Lemma 2.2 with convolutional sums being replaced by 'matrix-by-vector' multiplications. □

Corollary 2.1: *The block matrix in (10 a) is full rank and hence the dimension, $2(\mu + 1) - \gamma_+$, of the vector $\underline{\theta}$ of Theorem 2.2 defines the number of degrees of freedom in the choice of the eigenfunctions of causal NECCs.*

Proof: Let C_1 and C_2 be the matrices formed, respectively, out of the first μ and the last $\mu + 1$ columns of C_{D_+}. Then it is easy to show that the block matrix in (10 a) can only be rank deficient if two non-zero vectors, $\underline{u}_1$ and $\underline{u}_2$, exist such that $C_1\underline{u}_1 = \underline{0}$, $\underline{u}_2 = -C_2\underline{u}_1$ and $\underline{u}_2 = C_2\underline{u}_1$. But these conditions can only hold true if $\underline{u}_1 = \underline{u}_2 = \underline{0}$, because by construction C_2 has an upper triangular striped form and is full rank. □

Theorem 2.2 gives a complete characterization of the class of 2×2 causal NECC and the section below gives a numerical illustration of the salient features of the theorem.

2.3. *A 2 × 2 numerical example*

To facilitate comparisons between the present NECC work, and the earlier AECC work we shall base our numerical illustrations on the same transfer function matrix, $G(z) = N(z)/d(z)$ considered by Cloud and Kouvaritakis (1987). This is given in terms of the vectors $\underline{n}_{11}$, $\underline{n}_{12}$, $\underline{n}_{21}$, $\underline{n}_{22}$, and $\underline{d}$ of the coefficients of the numerator and common denominator polynomials of $G(z)$ as:

$$\begin{aligned}
\underline{n}_{11} &= [0, 0{\cdot}768, -2{\cdot}222, 1{\cdot}129, 2{\cdot}308, -2{\cdot}464, -0{\cdot}5, 1{\cdot}371, -0{\cdot}174, -0{\cdot}315,\\
&\qquad 0{\cdot}079, 0{\cdot}03, -0{\cdot}009, -0{\cdot}001, 0{\cdot}0002, 0{\cdot}000003]\\
\underline{n}_{12} &= [0, 0{\cdot}111, -0{\cdot}083, -0{\cdot}431, 0{\cdot}447, 0{\cdot}429, -0{\cdot}607, -0{\cdot}09, 0{\cdot}304, -0{\cdot}037,\\
&\qquad -0{\cdot}06, 0{\cdot}013, 0{\cdot}004, -0{\cdot}001, -0{\cdot}0001, 0{\cdot}00002]\\
\underline{n}_{21} &= [0, -0{\cdot}126, 0{\cdot}376, -0{\cdot}015, -0{\cdot}815, 0{\cdot}445, 0{\cdot}628, -0{\cdot}463, -0{\cdot}222, 0{\cdot}189,\\
&\qquad 0{\cdot}037, -0{\cdot}0328, -0{\cdot}0022, 0{\cdot}002, -0{\cdot}0001]\\
\underline{n}_{22} &= [0, 0{\cdot}499, -1{\cdot}287, 0{\cdot}415, 1{\cdot}42, -1{\cdot}092, -0{\cdot}477, 0{\cdot}635, 0{\cdot}006, -0{\cdot}156,\\
&\qquad 0{\cdot}024, 0{\cdot}018, -0{\cdot}004, -0{\cdot}001, 0{\cdot}0001]\\
\underline{d} &= [1, -4{\cdot}546, 7{\cdot}249, -2{\cdot}8381, -4{\cdot}825, 5{\cdot}628, -0{\cdot}629, -1{\cdot}914, 0{\cdot}86,\\
&\qquad -0{\cdot}158, -0{\cdot}17, 0{\cdot}017, 0{\cdot}009, -0{\cdot}002, 0{\cdot}0001]
\end{aligned}$$

For this example, the discriminant of the characteristic equation of $N(z)$ has two unstable branch points (i.e. $\gamma_+ = 1$) which are given by the roots of

$$D_+(z) = (z - 11{\cdot}986)(z - 1{\cdot}7682)$$

and the corresponding expansion for $z^{-(\gamma+)}\sqrt{D_+(z)}$ is given as

$$-0{\cdot}0001z^9 - 0{\cdot}0003z^8 - 0{\cdot}0006z^7 - 0{\cdot}0012z^6 - 0{\cdot}0028z^5 - 0{\cdot}0065z^4 - 0{\cdot}016z^3 - 0{\cdot}0434z^2 - 0{\cdot}1338z - 1{\cdot}4938 + 4{\cdot}6037z^{-1}$$

Note that we have implicitly chosen $\mu = 9$; this choice is dictated by the lengths of the causal and anti-causal components of the Laurent expansions of the eigenvectors and dual eigenvectors of $N(z)$ shown in Fig. 1.

The purpose of this example is to show that it is possible to use the results of Theorem 2.2 in order to get controllers that are both commutative and causal; at this stage we are not interested in the use of these controllers for the purpose of compensating the characteristic loci of $G(z)$. Hence $f_1(z)$ and $f_2(z)$ can be chosen arbitrarily and here we select these to be the polynomials (in z^{-1}) formed out of the first $\mu = 9$ and $\mu - \gamma_+ = 8$ terms of the Maclaurin expansions of $1/(1 - 0{\cdot}4z^{-1})$ and $1/(1 - 0{\cdot}2z^{-1})$, respectively. For this choice, the sequences of the coefficients for the controller eigenfunctions $k_1(z)$ and $k_2(z)$, as computed from (10) of Theorem 2.2 are shown in Fig. 2. The sequences of the coefficients of the elements of the resulting NECC, $K(z)$, are shown in Fig. 3, from which it can be seen clearly that $K(z)$ is causal; the coefficients with a

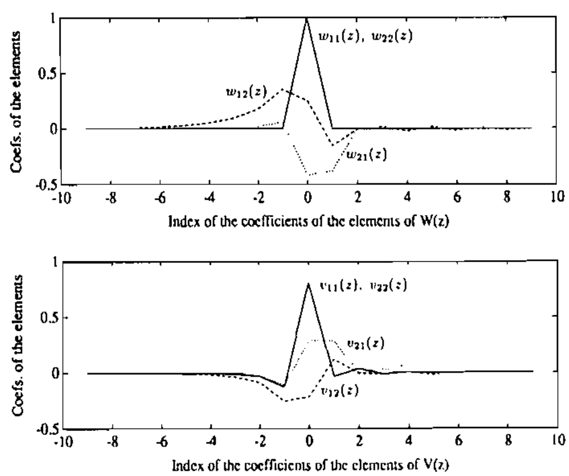

Figure 1. Coefficients of the bi-causal expansions of the elements of $W(z)$ and $V(z)$.

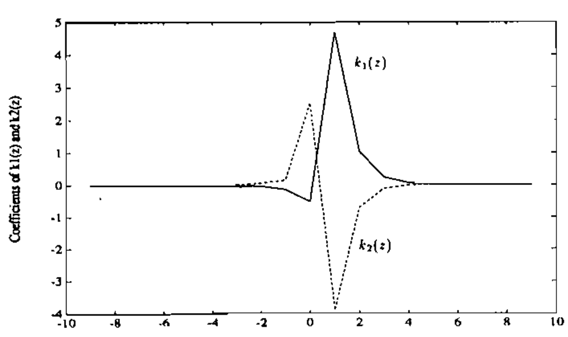

Figure 2. Coefficients of the bi-causal expansions of $k_1(z)$ and $k_2(z)$.

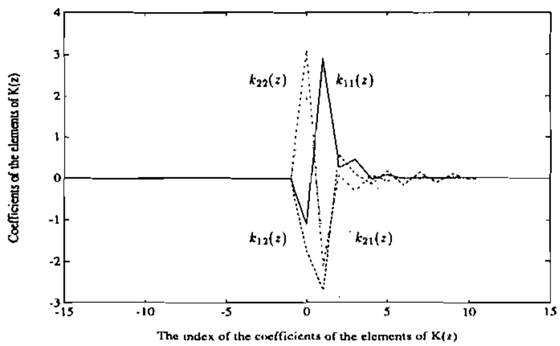

Figure 3. Coefficients of the bi-causal expansions of the elements of $K(z)$.

negative index (i.e. the coefficients of positive powers of z) are all zero. That the controller is also (nearly) commutative can be checked by evaluating the percentage commutativity index $\eta = 100\sigma_{\max}[G(z)K(z) - K(z)G(z)]/\sigma_{\max}[G(z)K(z)]$ for all values of z on the unit circle. For $\mu = 9$, detailed calculation shows that η ranges from 0·53% to 2·14%. These values are small and therefore very satisfactory, but can be made smaller still for larger values of μ; $\mu = 30$ for example leads to values of η which are of the order of 10^{-3}%!

In conclusion, therefore, for the particular choice of $f_1(z)$ and $f_2(z)$, Theorem 2.2 produced a controller that is both causal and commutative. Our choice of $f_1(z)$ and $f_2(z)$ was arbitrary, but in a design situation the coefficients of these two polynomials would characterize the available degrees of freedom for the 2×2 case. The section below considers the extension of Theorem 2.2 to the general $m \times m$ case.

2.4. *The characterization of causal exact commutative controllers for the $m \times m$ case*

In § 2.1 it was argued that if the eigenfunctions of a commutative controller were chosen to be $f(g_i(z))$, where $f(\cdot)$ is any function which is analytic inside the unit disc, then such a controller would be causal. This observation is not restricted to the 2×2 case; thus we know that the class of causal NECCs is not empty for the 2×2 as well as the general $m \times m$ case. However, unlike the 2×2 case where it is possible to give a characterization of the whole class of causal NECCs in terms of the individual elements of $N(z)$, in the general case one has to adopt a more indirect approach. In particular, it is easy to show that the p, q element of a commutative controller with eigenfunctions $k_i(z)$, $i = 1, 2, \ldots, m$ is given by

$$K_{p,q}(z) = \sum_{i=1}^{m} W_{p,i}(z)V_{i,q}(z)k_i(z) \tag{11}$$

where $W_{p,i}(z)$ and $V_{i,q}(z)$ denote the p, i and i, q elements of $W(z)$ and $V(z)$, respectively. Hence, using the bi-causal series expansions for $W(z)$ and $V(z)$ of (4), and defining finite Laurent expansions for $k_i(z)$ in a manner analoguous to that done for $k_1(z)$ and $k_2(z)$ in Theorem 2.2, it is possible to derive a bi-causal expansion for $K_{p,q}(z)$, by performing the convolutional sums implied by the right-hand side of (11). In more compact form, this can be summarized by writing, $\underline{K}_{p,q}$, the vector of coefficients of the expansion of $K_{p,q}(z)$ as

$$\underline{K}_{p,q} = \sum_{i=1}^{m} C_{W_{pi}} C_{V_{iq}} \underline{k}_i \tag{12}$$

where $\underline{k}_i$ is the vector of coefficients of the expansion of $k_i(z)$, and $C_{W_{pi}}$ and $C_{V_{iq}}$ are both lower triangular striped Toeplitz matrices formed out of the coefficients of the bi-causal expansions for $W_{pi}(z)$ and $V_{iq}(z)$, respectively. For simplicity and without loss of generality it is assumed that all the bi-causal expansions involved in the above have μ anti-causal terms and $\mu + 1$ causal terms. As a consequence, the dimensions of $\underline{k}_i$, $C_{W_{pi}}$, and $C_{V_{iq}}$ are $(2\mu + 1) \times 1$, $(4\mu + 1) \times (2\mu + 1)$, and $(6\mu + 1) \times (4\mu + 1)$. Clearly, the first 3μ elements of $\underline{K}_{p,q}$ correspond to the coefficients of positive powers in the expansion of $K_{p,q}(z)$ and thus we are led to the following characterization of causal NECCs for the general $m \times m$ case.

Theorem 2.3: *Let $C^+{}_{WV_{piq}}$ denote the matrix formed out of the first 3μ rows of $C_{W_{pi}} C_{V_{iq}}$ and let $k = [\underline{k}_1^t, \underline{k}_2^t, \ldots, \underline{k}_m^t]^t$, where $\underline{k}_i$ denotes the vector of coefficients of the bi-causal expansion of $\underline{k}_i(z)$. Then, (to within truncation errors) the NECC with eigenfunction $k_i(z)$, $i = 1, 2, \ldots, m$ will be causal if, and only if*

$$M\underline{k} = \underline{0}_{3\mu m^2}; \quad \text{with } M = \begin{bmatrix} C^+_{WV_{111}} & C^+_{WV_{121}} & \cdots & C^+_{WV_{1m1}} \\ C^+_{WV_{211}} & C^+_{WV_{221}} & \cdots & C^+_{WV_{2m1}} \\ \vdots & \vdots & & \vdots \\ C^+_{WV_{m11}} & C^+_{WV_{m21}} & \cdots & C^+_{WV_{mm1}} \\ C^+_{WV_{112}} & C^+_{WV_{122}} & \cdots & C^+_{WV_{1m2}} \\ \vdots & \vdots & & \vdots \\ C^+_{WV_{m1m}} & C^+_{WV_{m2m}} & \cdots & C^+_{WV_{mmm}} \end{bmatrix} \tag{13}$$

Proof: Partition the vector $M\underline{k}$ into m blocks of μ elements each, starting with the first μ elements and finishing with the last μ elements. Then, by construction, the 1st, 2nd, ..., mth, $(m + 1)$th, ..., m^2th blocks define the vectors of anti-causal coefficients of $K_{11}(z)$, $K_{21}(z)$, ..., $K_{m1}(z)$, $K_{12}(z)$, ..., $K_{mm}(z)$, respectively. Clearly if $K(z)$ is to be causal, all these blocks must be zero. The sufficiency is obvious. □

It is possible to show that (13 *a*) will have non-trivial solutions as stated in the corollary below.

Corollary 2.2: *To within truncation errors, the matrix M of Theorem 2.2 is always rank deficient, and its nullity v is greater than or equal to $\mu + 1$. Therefore, there exists a matrix $Y°$ of dimension $(2\mu + 1)m \times v$ such that*

$$MY° = 0_{3\mu m^2, v} \quad \text{and} \quad [Y°]^* Y° = I_v \tag{14}$$

Furthermore, if $\underline{k}$ denotes the vector of coefficients of bi-causal power series expansions of some functions $k_i(z)$, $i = 1, 2, \ldots, m$, then (to within truncation errors) the NECC with $k_i(z)$ as eigenfunctions will be causal if, and only if,

$$\underline{k} = Y°\theta \tag{15}$$

Proof: The p, q element of the commutative controller with eigenfunctions $k_i(z) = z^{-j}$, $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, \mu$ is given as

$$K_{p,q}(z) = \sum_{i=1}^{m} W_{p,i}(z)V_{i,q}(z)z^{-j} = z^{-j}\delta_{p,q} \tag{16}$$

where $\delta_{p,q}$ is one for $p = q$ and zero otherwise. This follows directly from the fact that $W(z)V(z) = I$. Thus such a controller would clearly be causal and would therefore lead to a vector $\underline{k}$, formed out of the coefficients of the $k_i(z)$, which would lie (to within truncation errors) in the kernel of M. The vector block that corresponds to each $k_i(z)$ would have the form of the $(j+1)$th column vector of the matrix E of (10) and would thus, for $j = 0, 1, \ldots, \mu$, lead to a set of linearly independent vectors, each of which would lie in the kernel of M. Clearly, then, the kernel of M is non-empty and has a rank defect, ν, which is at least $\mu + 1$. [In fact numerical experimentation shows that the dimension of the kernel is of order $m(\mu + 1) - \gamma_+$. This is consistent with Corollary 2.1 which states that, for the 2×2 case, the rank of the block matrix in (10), and hence, equivalently, the rank of Y^0, is $2(\mu + 1) - \gamma_+$]. Thus, given the singular value decomposition of M, $M = X\Sigma Y^*$, then ν of the input principal directions, namely the column vectors of Y, will correspond to zero singular values and hence will form the matrix Y^0 which satisfies the conditions of the theorem. □

Theorem 2.3 gives a complete characterization of the degrees of freedom available in the choice of NECC which are causal. The next section gives a brief description of how these degrees of freedom can be given up for the purposes of design.

3. The design algorithm

Under commutativity, the eigenvalues of $G(z)$ and $K(z)$ that share a common eigenvector, $w_i(z)$, will multiply to give the eigenvalues of $Q(z) = G(z)K(z)$ as

$$q_i(z) = g_i(z)k_i(z) = g_i(z)\underline{\phi}^{\mathrm{T}}(z)\underline{k}_i \quad \text{where} \quad \underline{\phi}(z) = [z^{\mu}, z^{\mu-1}, \ldots, z^{-\mu}]^{\mathrm{T}} \tag{17}$$

where use has been made of the bi-causal expansion of $k_i(z)$. The above will be true for every $i = 1, 2, \ldots, m$ and thus collecting all the $q_i(z)$ as elements of a vector $q(z)$ we may write

$$\left.\begin{aligned} \underline{q}(z) &= \Lambda_G(z)\Phi(z)\underline{k} = \Lambda_G(z)\Phi(z)Y^0\underline{\theta} \\ \Phi(z) &= \mathrm{diag}[\underline{\phi}^{\mathrm{T}}(z), \underline{\phi}^{\mathrm{T}}(z), \ldots, \underline{\phi}^{\mathrm{T}}(z)]) \end{aligned}\right\} \tag{18}$$

where $\Phi(z)$ is an $m \times (2\mu + 1)m$ matrix. Equation (18) can be applied at any desired set of values of z on the unit circle, say $z_1, z_2, \ldots, z_n$ to give the overall equation

$$\left.\begin{aligned} &\underline{q}_{\text{target}} = \Psi\theta \\ &\text{where} \\ &\underline{q}_{\text{target}} = \begin{bmatrix} \underline{q}(z_1) \\ \underline{q}(z_2) \\ \vdots \\ \underline{q}^{\mathrm{t}}(z_n) \end{bmatrix}; \quad \Psi = \begin{bmatrix} \Lambda_G(z_1)\Phi(z_1)Y^0 \\ \Lambda_G(z_2)\Phi(z_2)Y^0 \\ \vdots \\ \Lambda_G(z_n)\Phi(z_n)Y^0 \end{bmatrix} \end{aligned}\right\} \tag{19 a, b}$$

So, in theory, one can specify any arbitrary target vector $\underline{q}$ and then solve for the vector of degrees of freedom, $\underline{\theta}$, that will attain this target. However, in general (19 a) implies $2mn$ real simultaneous equations in the v elements of $\underline{\theta}$, and this imposes a limit on the number of frequency points n, if an exact solution for $\underline{\theta}$ were to exist. Too small a value for n, on the other hand, would in general cause aliasing difficulties, which would manifest themselves in inter-frequency errors: $q(z)$ would assume the correct values at the preselected frequencies z_i, $i = 1, 2, \ldots, n$, but would be in error for other points on the unit circle. To avoid this situation, n must be chosen to be greater than, or equal to μ_q, the maximum number of causal or anti-causal terms in the Laurent expansions of $q_i(z)$ for $i = 1, 2, \ldots, m$; although the $q_i(z)$ may not be known explicitly, the coefficients of their Laurent expansions can be computed from the knowledge of the frequency responses of $q_i(z)$, using an inverse discrete Fourier transform procedure (as that described in Theorem 2.1). Therefore there will exist $2m\mu_q$ linear simultaneous equations in ρ variables, and since v is of the order of $m\mu$, in general for an arbitrary set of target frequency responses $q_i(z_j)$ there will not exist enough degrees of freedom. A practical solution to this problem is proposed in the theorem below.

Theorem 3.1: *The vector $\underline{k}$ of coefficients of bi-causal expansions of the eigenfunctions $k_i(z)$, $i = 1, 2, \ldots, m$ of the causal NECC, $K(z)$, that minimizes the L_2-norm of the error between the frequency response of the eigenfunctions of $G(z)K(z)$ and the frequency response of a prescribed set of characteristic transfer functions $q_i(z)$ is given by*

$$\underline{k} = Y^0\underline{\theta}; \quad \underline{\theta} = (\Re[\Psi^* \Psi])^{-1}\Re[\Psi^* q_{\text{target}}] \tag{20}$$

where Ψ and $\underline{q}_{\text{target}}$ *are as defined in equation* (19) *for* $z_i = \exp(\mathrm{j}i\pi/(n-1))$, $i = 0, 1, \ldots, n-1$, *and* $n \geqslant \mu_q$, *where* μ_q *denotes the maximum number of causal or anti-causal coefficients in the Laurent expansions of the target* $q_i(z)$ *over all* i.

Proof: Under the assumption that $n \geqslant \mu_q$, to within truncation errors, the frequency sampled at the points z_i of the theorem uniquely define the Laurent expansion, and therefore the frequency response itself of the prespecified targets $q_i(z)$. Then the minimization of the L_2-norm of the error between the targets $q_i(z)$ and the eigenfunctions of $G(z)K(z)$ given by $g_i(z)k_i(z)$ is equivalent to the minimization of the cost

$$J = \|\underline{q}_{\text{target}} - \Psi\underline{\theta}\|_2 \tag{21}$$

Setting the derivative of J^2 with respect to $\underline{\theta}$ equal to zero, readily leads to the optimal choice of $\underline{\theta}$ given in (20). □

The implied design procedure will lead to controllers that are commutative (to within truncation errors) at all frequencies but which cannot exactly attain any arbitrary target frequency responses. Despite this limitation, it will be shown in the next section, by means of a design study, Example 4.1, that the NECCs of Theorem 3.1 offer a significant improvement on the approximately exact commutative controllers proposed earlier by Cloud and Kouvaritakis (1987). How close the NECC of the theorem above can get to the target frequency responses depends on the choice of these targets. From a design point of view the overall aim is to adjust the gain/phase characteristics of the characteristic loci of $G(z)$ so as to improve the compensated system's relative stability margins. It follows therefore that the definition of target frequency responses is by no means unique, and that there exists freedom which can be exploited in such a manner that the NECC of Theorem 3.1 will lead to insignificantly small optimal cost values J. That this can be done is also illustrated in § 4, where a small modification of the targets of Example 4.1 (which does not affect the relative stability margins) results in a very accurate NECC; the relevant controller attains the targets to within an accuracy of better than 2% over all frequencies. The choice of attainable targets falls beyond the scope of this paper. Here we simply point out that, normally, achieving certain gain/phase characteristics is critical over a limited range of frequencies (the frequencies over which the characteristic loci come close to the critical point) and therefore weighting factors can be introduced into the cost of (21) in order to improve the NECC's accuracy over the desired frequency range at the expense of frequencies where the choice of targets is rather arbitrary.

4. Design studies

AECCs offer a very significant advantage over ACCs, as was illustrated by means of a design study by Cloud and Kouvaritakis (1987). To illustrate the superiority of NECCs over AECCs here, we use the same design study. The model under consideration is that described in § 2.3. The characteristic loci of this model are shown in Fig. 4 from which it can be seen that the relative stability margins due to the second locus are poor; in fact, the uncompensated system clearly would be unstable under unity feedback. With a view to improving the gain/phase characteristics of $g_2(z)$ and decreasing the disparity between the d.c. gains of the two loci, Cloud and Kouvaritakis (1987) proposed as targets

$$q_1(z) = k_1(z)g_1(z) = 0{\cdot}81g_1(z); \quad q_2(z) = k_2(z)g_2(z) = \frac{1{\cdot}36(1 - 0{\cdot}63z^{-1})}{1 + 0{\cdot}0855z^{-1}}g_2(z) \tag{22}$$

The resulting target frequency responses are shown in Fig. 5 (as solid curves) and can be seen to have satisfactory relative stability margins.

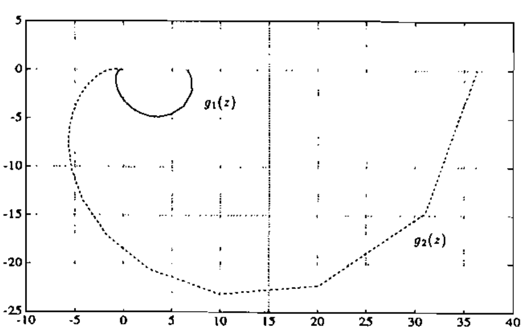

Figure 4. Characteristic loci of $G(z)$.

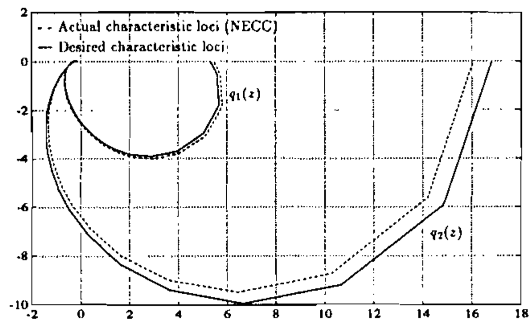

Figure 5. Characteristic loci of $Q(z)$.

For this example the discriminant of the characteristic equation of $N(z)$ has two unstable roots (one at 11·99, the other at 1·76) so that the given $G(z)$ has unstable branch points. As a consequence, the resulting AECC with the $k_1(z)$ and $k_2(z)$ implied by (22) cannot be exactly commutative with $G(z)$ so that the gain/phase adjustments of $g_1(z)$ and $g_2(z)$ are not expected to be exact. This indeed is shown to be the case by the dashed curves of Fig. 6, which depict the percentage error in the attainment of the target frequency responses when $z = \exp(j\phi)$, as ϕ varies from 0 to π.

To overcome this difficulty, we now propose to use an NECC which deploys near exact eigenvector approximations through the use of bi-causal expansions. The sequence of coefficients for the elements of $W(z)$ and $V(z)$ are shown in Figs 1(a), 1(b) and from these it is seen that there are at the most nine causal and anti-causal terms in both $W(z)$ and $V(z)$; thus μ is chosen to be nine. To ensure the causality of the resulting NECC we need to constrain the vector $\underline{k}$ of the coefficients of the expansion of the eigenfunctions $k_1(z)$ and $k_2(z)$ to lie in the kernel of the matrix M of (13) which, for this example, has $3\mu m^2 = 108$ rows and $(2\mu + 1)m = 38$ columns. Figure 7 shows the singular values of M and confirms that the dimension of the kernel of M is $2(\mu + 1) - \gamma_+ = 19$. The

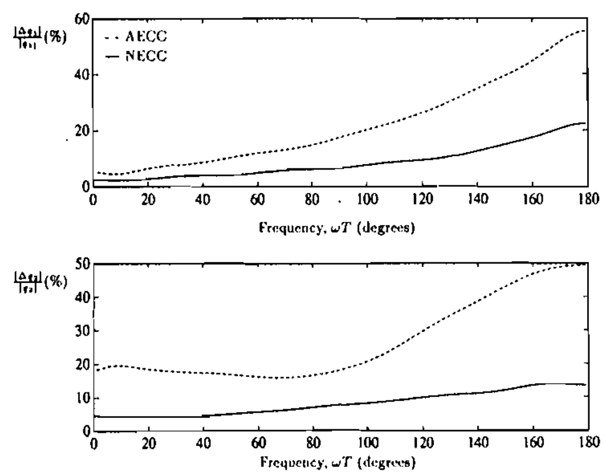

Figure 6. Comparison between NECC and AECC.

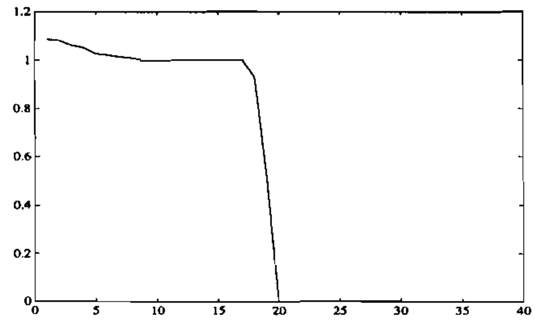

Figure 7. Singular values of M.

matrix representation of the kernel Y^0 can be formed out of all the input principal directions of M that correspond to the nearly zero singular values and $\underline{k}$ then is constrained to lie in the linear span of Y^0. The degrees of freedom in the choice of $\underline{k}$ are then given up with a view to minimizing the cost of Theorem 3.1, formed using the same target $q_i(z)$ as those used in Cloud and Kouvaritakis (1987), which targets were sampled at $n = 90$ points; as required, n was taken to be greater than (or equal to) the number of significant causal/anti-causal terms in the Laurent expansions of $q_i(z)$, which for this example is given by $\mu_q = 75$. (It is noted that the large value of μ_q is due to the position of the transfer function poles—namely the roots of the common least denominator of $G(z)$, $d(z)$. Had we used $d(z)q_i(z)$ as targets for the eigenfunctions of $N(z)K(z)$, then the corresponding value of μ would have been $\mu_{\mathrm{dq}} = 13$). The characteristic loci of $G(z)K(z)$, where $K(z)$ denotes the resulting NECC, are shown (dashed lines) together with the target frequency responses (solid curves) in Fig. 5. Although the targets have not been attained exactly, the NECC has in fact effected the desired improvement of the relative stability margins of the characteristic loci. More importantly, the percentage error of this operation which is depicted in Fig. 6 (in solid lines) can be seen to be considerably better than that achieved through an AECC; indeed the NECC accuracy is at least twice as good over all frequencies and about three to five times as good at high frequencies.

As explained in § 3, the accuracy of an NECC can be improved arbitrarily by an appropriate modification of the targets. To illustrate this claim, here we replace the $q_i(z)$ of (22) by

$$\left.\begin{aligned} q_1(z) &= k_1(z)g_1(z) = 0{\cdot}887\frac{(1 - 0{\cdot}68z^{-1})}{(1 - 0{\cdot}22z^{-1})}g_1(z) \\ &\text{and} \\ q_2(z) &= k_2(z)g_2(z) = \frac{0{\cdot}883(1 - 0{\cdot}63z^{-1})(1 + 0{\cdot}5122z^{-1})}{(1 + 0{\cdot}0855z^{-1})(1 + 0{\cdot}22z^{-1})}g_2(z) \end{aligned}\right\} \qquad (23)$$

The overall gain/phase characteristic of the new target characteristic loci, shown in Fig. 8, are similar to those of Fig. 5, and display almost identical relative stability margins; thus, from a design point of view, the targets of (23) would be as acceptable as those of (22). However for this set of targets the resulting

NECC (designed following the procedure described above) produce characteristic loci which are virtually indistinguishable from the frequency response of the targets; for this reason they have not been superimposed on the plots of Fig. 8. A more meaningful measure of the controller accuracy is the percentage error between desired and achieved frequency responses shown in Fig. 9; for the purposes of comparison, the figure depicts the results for both an NECC (solid curves) and the corresponding AECC (dashed curves). Clearly, the redefinition of targets had a beneficial effect on the accuracy of the AECC as well as on the NECC, but the NECC results now are a factor of about 10 or more better than those achieved by the AECC.

5. Conclusions

Commutativity provides a convenient means for the systematic adjustment of the generalized Nyquist diagrams of a multivariable linear transfer function, $G(z)$. However, the practical implementation of commutative controllers is problematic due to the irrational nature of eigenvector functions. Earlier work proposed a simple but inexact solution which was based on real constant approximations to the frequency response of the eigenvector matrix of $G(z)$.

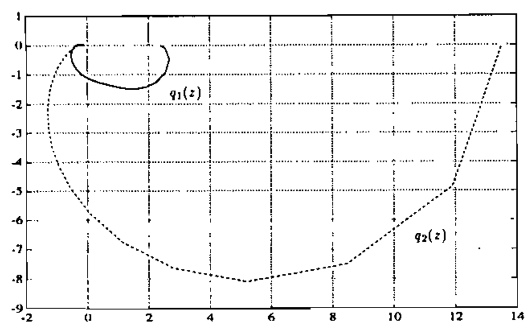

Figure 8. New targets for the characteristic loci of $Q(z)$.

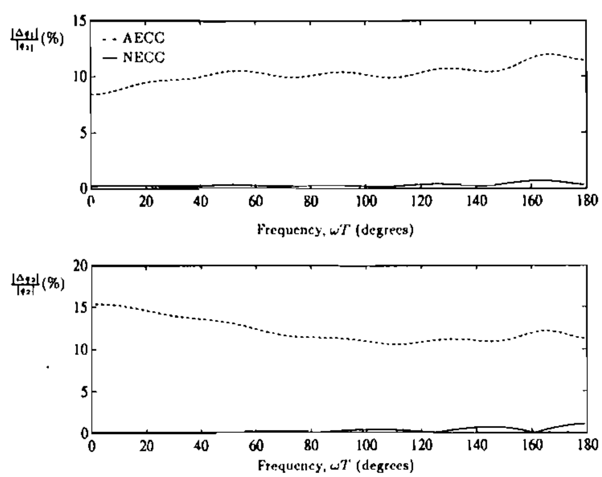

Figure 9. Comparison between NECC and AECC.

More recent work improved upon this by deploying causal power series eigenvector approximations, and yielded nearly exact results for the case of stable branch points. To handle the general case, in this paper we proposed a procedure which uses bi-causal approximations for the eigenvector matrix of $G(z)$ and thus results in controllers which can be made to commute (under multiplication) with $G(z)$ over all frequencies to within any desired degree of accuracy. For an arbitrary choice of controller eigenfunctions, the resulting commutative controllers would be anti-causal. In this paper we have shown that it is possible to achieve causality be constraining the choice of controller eigenfunctions. Furthermore it was proved that there still remain enough degrees of freedom to be exploited for the purposes of adjusting the gain/phase characteristics of the characteristic loci of $G(z)$. Finally, we demonstrated by means of a design study that the results obtained by such commutative controllers are far superior to those possible through the use of the forms of approximately commutative controllers proposed earlier.

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References

Cloud, D. J., and Kouvaritakis, B., 1987, Commutative controllers revisited: parallel computation, a new lease of life. *International Journal of Control*, **45**, 1335–1370.

Kouvaritakis, B., and Rossiter, J. A., 1991, Bicausal representations and multivariable generalized predictive control. *Automatica*, **27**, 819–828.

Kouvaritakis, B., Rossiter, J. A., and Trimboli, M. S., 1990, H_2- and H_∞-approximations for eigenvalues/vector functions of transfer matrices. *International Journal of Control*, **5**, 1015–1049.

MacFarlane, A. G. J., and Belletruttic, J. J., 1973, The characteristic locus design method. *Automatica*, **9**, 575–588.

MacFarlane, A. G. J., and Kouvaritakis, B., 1977, A design technique for linear multivariable feedback systems. *International Journal of Control*, **25**, 837–874.

MacFarlane, A. G. J., and Postlethwaite, I., 1977, Generalized Nyquist stability criterion and multivariable root loci. *International Journal of Control*, **25**, 81–127.

Murray, J. D., 1974, *Asymptotic Analysis* (Oxford, U.K.: Clarendon Press).

Rossiter, J. A., and Kouvaritakis, B., 1991, Causal eigenvector function approximations and the problem of scaling an algorithm. *International Journal of Control*, **53**, 509–525.