Fragility problem revisited: overview and reformulation

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Abstract: In a debate paper, Keel and Bhattacharyya have suggested, by means of simple examples taken from the open literature, that optimal and robust controllers can be fragile in the sense that a minute perturbation in the controller parameters can make the closed-loop system unstable. However, is it true that the optimal and robust controllers presented by Keel and Bhattacharyya are actually fragile? It is demonstrated that the particular parametric stability margin used by Keel and Bhattacharyya can be very conservative and to overcome this problem, two non-conservative measures of controller fragility are proposed. In addition, it will be shown that the examples in Keel and Bhattacharyya's paper are very special and the resulting fragility cannot be linked to the H_∞ optimisation but to non-appropriate H_∞ optimisation criterions and to bad choice of weights.

1 Introduction

In a debate paper [1], Keel and Bhattacharyya have suggested, by means of simple examples taken from the open literature, that optimum and robust controllers, designed by using H_2, H_∞, l_1 and μ formulations can be extremely fragile in the sense that a minute perturbation in the controller parameters can make the closed-loop system unstable. The measure of fragility used in [1] is the so-called relative parametric stability margin [2].

Different explanations for the fragility problem can be found in the literature. Mäkilä [3] has pointed out that the fragility problem was related to the controller realisation and also that, differently from the study carried out in [1], in a realistic situation, several control objectives are considered, leading therefore to non-fragile controllers. Faris *et al.* [4] examine Examples 3, 4 and 5 of [1] and present a procedure for assessing the fragility on the basis of the inherent robustness of the closed-loop system to perturbation in the physical parameters that make up implementation, using first- and second-order active RC filters in the implementation of continuous-time controllers and considering the effects of floating point erros in the implementation of digital controllers. More recently, Examples 1 and 2 of [1] have been revisited [5, 6]. In [5], these examples are considered under a different perspective, that is the fragility of repeated poles and zeros of the controller transfer function and in [6] the same examples are discussed from a robust control perspective. Another approach has been presented by Whidborne *et al.* [7], who proposed a controller fragility measure based on the closed-loop pole sensitivity, which can be used to find an optimal state-space realisation for the controller in order to reduce the sensitivity of the closed-loop poles. The fragility problem has also been related to constrained optimisation in [8], where some conditions which may lead to the controller

fragility are presented, for example, degeneracy of the roots of the closed-loop characteristic polynomial. The main contribution of [1] appears to be the establishment of another research topic, namely the design of robust controllers that are also non-fragile (see [9–15] and the references therein)

In spite of all the works listed in the previous paragraph, some questions still remain to be answered. Is it true that the optimal and robust controllers presented in [1] are actually so fragile? More importantly, is it true that the controllers obtained as solutions of the simple optimisation criteria presented in [1] are necessarily fragile? In this paper, these questions are answered and it is demonstrated that the particular stability margin used by Keel and Bhattacharyya can be very conservative and to overcome this problem, two non-conservative measures, based on necessary and sufficient conditions, are proposed here. In addition, it will be shown that the examples presented in [1] are very special and the resulting fragility cannot be associated with H_∞ optimisation but to non-appropriate H_∞ optimisation criterions and to bad choice of weights.

This paper is organised as follows: in section 2, the relative parametric stability margin is reviewed, and an example that suggests the conservativeness of this measure is presented. In section 3, two nonconservative measures of controller fragility are proposed and a comparison between the relative parametric stability margin and the two nonconservative measures introduced in this paper is drawn. In section 4, the examples used in [1] to label H_∞ controllers as fragile are re-examined. Finally, conclusions are drawn in section 5.

2 Relative parametric stability margin

2.1 Definition

Consider a closed-loop system with unit negative feedback, where

$$G(s) = \frac{n_G(s)}{d_G(s)} = \frac{\sum_{i=0}^{m_1} \alpha_i s^{m_1-i}}{\sum_{j=0}^{n_1} \beta_j s^{n_1-j}},$$

$$K_0(s) = \frac{\sum_{k=0}^{m_2} \epsilon_k s^{m_2-k}}{\sum_{q=0}^{n_2} \phi_q s^{n_2-q}} \quad (1)$$

are the transfer function representations of the plant and the nominal controller, and assume that $K_0(s)$ stabilises $G(s)$. Let $\underline{p}^0 = [\begin{matrix} p_1^0 & p_2^0 & \cdots & p_l^0 \end{matrix}]^T (l \le m_2 + n_2 + 2)$ denote a vector formed with the parameters of $K_0(s)$ whose elements belong to the set $\mathcal{P} = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_{m_2}, \phi_0, \phi_1, \ldots, \phi_{n_2}\}$, being chosen among those parameters, which are subject to perturbation, and define the parameter vector

$$\underline{p} = \underline{p}^0 + \Delta\underline{p} = [\begin{matrix} p_1^0 + \Delta p_1 & p_2^0 + \Delta p_2 & \cdots & p_l^0 + \Delta p_l \end{matrix}]^T \quad (2)$$

Assuming that the plant parameters are kept fixed, then it is easy to check that, to each vector $\underline{p}$, there corresponds a closed-loop characteristic polynomial $\delta(s, \underline{p})$ that can be written as [2]

$$\delta(s, \underline{p}) = \delta(s, \underline{p}^0 + \Delta\underline{p}) = \delta(s, \underline{p}^0) + \sum_{i=1}^{l} a_i(s)\Delta p_i \quad (3)$$

where $a_i(s)$ are polynomials whose coefficients do not depend on $\underline{p}$. According to the boundary crossing theorem [2], when the coefficients of a fixed degree polynomial are subjected to change, in order to its zeros to move from the open left half-plane to the right half-plane, they must cross $j\omega$-axis, $\omega \in \mathbb{R}_+$. Substituting $s = j\omega$ in (3), and for each $\omega \in \mathbb{R}_+$, writing $a_k(j\omega) = a_{kR}(j\omega) + ja_{kI}(j\omega)$ and $\delta(j\omega, \underline{p}^0) = \delta_R(j\omega, \underline{p}^0) + j\delta_I(j\omega, \underline{p}^0)$ where $a_{kR}(j\omega)$, $a_{kI}(j\omega)$, $\delta_R(j\omega, \underline{p}^0)$, $\delta_I(j\omega, \underline{p}^0) \in \mathbb{R}$, then (3) can be re-written as

$$A(j\omega)\Delta\underline{p} = -\underline{y}(j\omega), \quad (4)$$

$$A(j\omega) = \begin{bmatrix} a_{1R}(j\omega) & a_{2R}(j\omega) & \cdots & a_{lR}(j\omega) \\ a_{1I}(j\omega) & a_{2I}(j\omega) & \cdots & a_{lI}(j\omega) \end{bmatrix},$$

$$\Delta\underline{p} = \begin{bmatrix} \Delta p_1 \\ \Delta p_2 \\ \vdots \\ \Delta p_l \end{bmatrix}, \quad \underline{y}(j\omega) = \begin{bmatrix} \delta_R(j\omega, \underline{p}^0) \\ \delta_I(j\omega, \underline{p}^0) \end{bmatrix} \quad (5)$$

Definition 1: Let $\Delta\underline{p}$ defined according to (4) and (5). The parametric stability margin ρ is defined as the smallest size of $\|\Delta\underline{p}\|_2$ for all $\omega \in \mathbb{R}_+$ that makes the closed-loop system unstable, that is

$$\rho = \min_{\omega \in \mathbb{R}_+} \|\Delta\underline{p}(j\omega)\|_2 \quad (6)$$

It is worth remarking that if for some frequency ω_k, the rank of $A(j\omega_k)$ is equal to one, then (4) may not have a solution. In this case $\rho(j\omega_k) = \infty$. In addition, notice that the loss of stability can also be due to the loss of degree of the closed-loop characteristic polynomial. Therefore, the value of $\|\Delta\underline{p}\|_2 (\rho_d)$, which makes the leading coefficient of the closed-loop polynomial equal to zero must be taken into account in the computation of ρ, that is, ρ is chosen between the smallest value of (6) and ρ_d. Finally, following the usual practice for plant uncertainty analysis, the relative parametric stability margin can be used, being defined as

$$\bar{\rho} = \frac{\rho}{\|\underline{p}^0\|_2} \quad (7)$$

Remark 1: It is important to notice that the state-space representations associated directly with transfer functions are the so-called controller and observer realisations [16]. However, in practice, these forms are not usually used in the controller implementation with analogue components; the so-called parallel and cascade realisations are preferable from the sensitivity standpoint. This leads to the following question:

what are the largest perturbations in the coefficients that can be tolerated in the actual controller implementation? This is a difficult question and has been addressed in [13], for example. In this paper, the fragility problem is approached according to its original formulation [1, 2], and the issue of controller realisation will not be addressed directly. □

2.2 Critical appraisal to relative parametric stability margin as a measure of controller fragility

It is clear from (6) that the parametric stability margin represents the largest stability hypersphere in the controller parameter space, being therefore a vectorwise measure. However, it is well known [17] that when uncertainties of a given polytopic nature are considered, this margin provides only sufficient conditions for stability that may be extremely conservative. In spite of it, the parametric stability margin has been used in [1] as a componentwise measure, that is, its value was deployed to give an allowed percent variation on each parameter of $\underline{p}^0$ such that the closed-loop system remains stable. This structure for the perturbations on the controller characterises a polytope (hyper-rectangle) in the controller parameter space and thus, it is expected that the controllers considered in [1] tolerate variations on their coefficients larger than those prescribed by the relative parametric stability margin.

Example 1: Consider the design of a robust controller for an electromagnetic suspension via μ-synthesis technique originally presented in [18] and re-examined in [1] (Example 4). The plant transfer function is

$$G(s) = \frac{-36.27}{s^3 + 45.69s^2 - 4480.9636s - 204\,735.226884}$$

and the designed controller has the following transfer function

$$K(s) = \frac{\epsilon_0^0 s^6 + \epsilon_1^0 s^5 + \epsilon_2^0 s^4 + \epsilon_3^0 s^3 + \epsilon_4^0 s^2 + \epsilon_5^0 s + \epsilon_6^0}{s^7 + \phi_1^0 s^6 + \phi_2^0 s^5 + \phi_3^0 s^4 + \phi_4^0 s^3 + \phi_5^0 s^2 + \phi_6^0 s + \phi_7^0}$$

where

$$\epsilon_0^0 = -5.220000000000000 \times 10^8,$$

$$\epsilon_1^0 = -1.190629800000000 \times 10^{11},$$

$$\epsilon_2^0 = -1.089211902480000 \times 10^{13},$$

$$\epsilon_3^0 = -5.104622252074320 \times 10^{14},$$

$$\epsilon_4^0 = -1.285270261841830 \times 10^{16},$$

$$\epsilon_5^0 = -1.629532689765926 \times 10^{17},$$

$$\epsilon_6^0 = -7.937217972339767 \times 10^{17},$$

$$\phi_1^0 = 1.468170000000000 \times 10^3,$$

$$\phi_2^0 = 8.153914724000001 \times 10^5,$$

$$\phi_3^0 = 2.268680248018680 \times 10^8,$$

$$\phi_4^0 = 1.818763428483511 \times 10^{10},$$

$$\phi_5^0 = 5.698409038920188 \times 10^{11},$$

$$\phi_6^0 = 6.284542925855980 \times 10^{12} \text{ and}$$

$$\phi_7^0 = 6.227740485023126 \times 10^{11}.$$

Defining as the nominal parameter vector $\underline{p}^0 = [\epsilon_0^0 \ \epsilon_1^0 \ \cdots \ \epsilon_6^0 \ \phi_1^0 \ \phi_2^0 \ \cdots \ \phi_7^0]^T$, then the parametric stability margin is

$\rho = 1.179386729005542 \times 10^3$ and the relative parametric stability margin is $\bar{\rho} = 1.455352715523672 \times 10^{-15}$. It has been concluded in [1] (see also Example 2 of [1]) that since $\bar{\rho} = 1.455352715523672 \times 10^{-15}$ then, according to Keel and Bhattacharyya's measure of fragility, this system can tolerate a percent change in all the controller coefficients of only $1.455352715523672 \times 10^{-13}\%$.

Let us now consider the details involved in the computation of ρ. The vector of parameter perturbation $\Delta\underline{p}$, which leads to the minute value of $\bar{\rho}$ obtained above, and the corresponding vector of percent changes in the coefficients of p^0, that is

$$\Delta\underline{p}(\%) = \left[\frac{\Delta\epsilon_0}{\epsilon_0^0} \frac{\Delta\epsilon_1}{\epsilon_1^0} \cdots \frac{\Delta\epsilon_6}{\epsilon_6^0} \frac{\Delta\phi_1}{\phi_1^0} \frac{\Delta\phi_2}{\phi_2^0} \cdots \frac{\Delta\phi_7}{\phi_7^0}\right]^T \times 100\%$$

are, respectively, given as

$$\Delta\underline{p} = \begin{bmatrix} -2.492463778436260 \times 10^{-5} \\ 7.069742475243460 \times 10^{-8} \\ 3.247151359604312 \times 10^{-11} \\ -9.210374124301137 \times 10^{-14} \\ -4.230349120239370 \times 10^{-17} \\ 1.199916288417147 \times 10^{-19} \\ 5.511247150856264 \times 10^{-23} \\ -1.179386638592689 \times 10^3 \\ -4.618019789340906 \times 10^{-1} \\ 1.536490503957842 \times 10^{-3} \\ 6.016299762268140 \times 10^{-7} \\ -2.001721056947192 \times 10^{-9} \\ -7.837961827881649 \times 10^{-13} \\ 2.607817737566834 \times 10^{-15} \end{bmatrix},$$

$$\Delta\underline{p}(\%) = \begin{bmatrix} 4.774834824590537 \times 10^{-12} \\ -5.937817510735461 \times 10^{-17} \\ -2.981193422704023 \times 10^{-22} \\ 1.804320411869537 \times 10^{-26} \\ 3.291408232053199 \times 10^{-31} \\ -7.363560706410304 \times 10^{-35} \\ -6.943550208728406 \times 10^{-39} \\ -80.33038671221240 \\ -5.663561547618787 \times 10^{-5} \\ 6.772618156744279 \times 10^{-10} \\ 3.307906717304374 \times 10^{-15} \\ -3.512771798716831 \times 10^{-19} \\ -1.247180888149330 \times 10^{-23} \\ 4.187421977261709 \times 10^{-25} \end{bmatrix} \quad (8)$$

Note that, although $\bar{\rho}$ is of the order of magnitude of $10^{-13}\%$, the required percent perturbation on p_8^0 (eighth element of $\Delta\underline{p}$ (%)), necessary to destabilise the feedback system, is about -80%. This shows that to make the closed-loop system unstable, all the perturbations on the controller coefficients should concentrate mainly in p_8^0, whereas the other parameters are subject to very small perturbations.

Example 1 suggests that if all perturbations on p_8^0 are limited to be $<80\%$, then the other controller coefficients would tolerate greater perturbations so that the feedback system remains stable. Therefore if one is interested in

obtaining the maximum percent variation in each controller parameter for closed-loop system stability, then it is necessary to take into account the perturbation structure, that is, it is necessary to define a parametric stability margin that has, as stability domain, a hyper-rectangle centered at the components of $\underline{p}^0$, the nominal controller parameter vector .

3 Non-conservative measures of controller fragility

Let the ith component of the parameter perturbation vector $\Delta\underline{p}$ introduced in (2) be defined as $|\Delta p_i| \le \tilde{\rho}|p_i^0|$, $i = 1, \ldots, l$. Then, the problem of defining a non-conservative measure of controller fragility can be stated as follows: find the smallest value of $\tilde{\rho}$ for which the closed-loop system becomes unstable or, equivalently, find the maximal hyper-rectangle, $\mathcal{H}(\tilde{\rho}, \underline{p}^0)$, centered at the nominal controller parameter vector $\underline{p}^0$ such that all vectors $\underline{p} \in \mathcal{H}(\tilde{\rho}, p^0)$ lead to controllers that stabilise the closed-loop system, where p_i, the ith component of $\underline{p}$, is as follows

$$p_i \in (p_i^0 - \tilde{\rho}|p_i^0|, \, p_i^0 + \tilde{\rho}|p_i^0|), \quad i = 1, 2, \ldots, l \quad (9)$$

3.1 Componentwise parametric stability margin using linear programming

Instead of solving the equation system (4), let us, for each frequency ω, formulate the following linear programming problem

$$\min \tilde{\rho} \quad (10)$$

subject to the following constraints:

(i) $\tilde{\rho} \ge 0$; (ii) $A(j\omega)\Delta\underline{p} = -\underline{y}(j\omega)$;

$$\text{(iii)} \quad \begin{bmatrix} -\tilde{\rho}|p_1^0| \\ -\tilde{\rho}|p_2^0| \\ \vdots \\ -\tilde{\rho}|p_l^0| \end{bmatrix} \le \Delta\underline{p} \le \begin{bmatrix} \tilde{\rho}|p_1^0| \\ \tilde{\rho}|p_2^0| \\ \vdots \\ \tilde{\rho}|p_l^0| \end{bmatrix} \quad (11)$$

Let $\tilde{\rho}^\star(\omega)$ denote the solution to the linear programming problem (10), at frequency ω. Therefore, the maximum allowed percent variation in the elements of $\underline{p}_0$ is given by

$$\rho_{\text{LP}} = \min_{\omega \in R_+} \tilde{\rho}^\star(\omega) \quad (12)$$

A similar formulation to the problem of finding the maximal polytope of perturbations in the plant, for discrete-time systems, has been presented in [17], where the search for the maximum allowed percent variation in the plant parameters also leads to the one-parameter optimisation problem given by (12), with the appropriate stability boundary region.

The non-conservativeness of the measure given in (12) is guaranteed by the following theorem.

Theorem 1: Let ρ_{LP} be given by (12). Then the hyper-rectangle $\mathcal{H}(\rho_{\text{LP}}, \underline{p}^0)$ is maximal in the class of all boxes subject to the admissible perturbations given by (9) and such that all vectors $\underline{p} \in \mathcal{H}(\rho_{\text{LP}}, \underline{p}^0)$, except at the boundary of $\mathcal{H}(\rho_{\text{LP}}, \underline{p}^0)$, represent controllers that stabilise the closed-loop system.

Proof: Let $\tilde{\rho} < \rho_{\text{LP}}$. Then, for all frequencies, (11.ii) is not satisfied with any perturbation vector $\Delta\underline{p}$ defined according to (11.iii), which implies that all closed-loop characteristic polynomials $\delta(s, \underline{p})$, where $\underline{p} \in \mathcal{H}(\tilde{\rho}, p^0)$, do not have any zero on the imaginary axis. According to the

boundary crossing theorem, all controllers associated with $\underline{p} \in \mathcal{H}(\tilde{\rho}, \underline{p}^0)$ stabilise the closed-loop system. □

The computation of ρ_{LP} can be carried out according to the following algorithm.

Algorithm 1:
Step 1. Choose a finite number N of frequency points ω_k $k = 1, \ldots, N$ and set $k = 1$.
Step 2. Set $\omega = \omega_k$ and formulate the linear programming problem (10) with the constraints imposed by (11). Find $\tilde{\rho}^\star(\omega_k)$ solution to the linear programming problem (10).
Step 3. Set $k = k + 1$ and go back to step 2 until $k = N$.
Step 4. Find $\rho_{\mathrm{LP}} = \min_k \tilde{\rho}^\star(\omega_k)$.

Remark 2: As for the computation of the parametric stability margin, it is also necessary to take into account the loss of degree of the closed-loop characteristic polynomial $\delta(s, \underline{p}^0 + \Delta\underline{p})$ in the computation of ρ_{LP}, that is, to find $\Delta\underline{p}$ that makes the leading coefficient of $\delta(s, \underline{p}^0 + \Delta\underline{p})$ equal to zero.

3.2 Componentwise parametric stability margin using generalised Kharitonov theorem

Another necessary and sufficient condition for the closed-loop stability of all systems whose coefficients of the numerator and denominator polynomial of the controller transfer function leads to parameter vectors in $\mathcal{H}(\tilde{\rho}, p^0)$, is provided by the Generalised Kharitonov theorem [2]. In order to do so, it is necessary to check if all the 32 generalised Kharitonov segments formed from $n_K(s)$, $d_K(s)$, $n_G(s)$ and $d_G(s)$ are Hurwitz. For completeness, the construction of these 32 polynomial segments will now be presented in detail. First, define $p_i = [p_i^0 - \tilde{\rho}|p_i^0|, p_i^0 + \tilde{\rho}|p_i^0|] = [p_i^{\min}, p_i^{\max}]$ and form all the four Kharitonov polynomials for $n_K(s)$ and $d_K(s)$, as follows

$$n_K^1(s) = p_{m_2+1}^{\min} + p_{m_2}^{\min}s + p_{m_2-1}^{\max}s^2 + p_{m_2-2}^{\max}s^3 + p_{m_2-3}^{\min}s^4$$
$$+ p_{m_2-4}^{\min}s^5 + p_{m_2-5}^{\max}s^6 + p_{m_2-6}^{\max}s^7 + \cdots$$

$$n_K^2(s) = p_{m_2+1}^{\min} + p_{m_2}^{\max}s + p_{m_2-1}^{\max}s^2 + p_{m_2-2}^{\min}s^3$$
$$+ p_{m_2-3}^{\min}s^4 + p_{m_2-4}^{\max}s^5 + p_{m_2-5}^{\max}s^6 + p_{m_2-6}^{\min}s^7 + \cdots$$

$$n_K^3(s) = p_{m_2+1}^{\max} + p_{m_2}^{\min}s + p_{m_2-1}^{\min}s^2 + p_{m_2-2}^{\max}s^3 + p_{m_2-3}^{\max}s^4$$
$$+ p_{m_2-4}^{\min}s^5 + p_{m_2-5}^{\min}s^6 + p_{m_2-6}^{\max}s^7 + \cdots$$

$$n_K^4(s) = p_{m_2+1}^{\max} + p_{m_2}^{\max}s + p_{m_2-1}^{\min}s^2 + p_{m_2-2}^{\min}s^3 + p_{m_2-3}^{\max}s^4$$
$$+ p_{m_2-4}^{\max}s^5 + p_{m_2-5}^{\min}s^6 + p_{m_2-6}^{\min}s^7 + \cdots$$

$$d_K^1(s) = p_l^{\min} + p_{l-1}^{\min}s + p_{l-2}^{\max}s^2 + p_{l-3}^{\max}s^3 + p_{l-4}^{\min}s^4 + p_{l-5}^{\min}s^5$$
$$+ p_{l-6}^{\max}s^6 + p_{l-7}^{\max}s^7 + \cdots$$

$$d_K^2(s) = p_l^{\min} + p_{l-1}^{\max}s + p_{l-2}^{\max}s^2 + p_{l-3}^{\min}s^3 + p_{l-4}^{\min}s^4 + p_{l-5}^{\max}s^5$$
$$+ p_{l-6}^{\max}s^6 + p_{l-7}^{\min}s^7 + \cdots$$

$$d_K^3(s) = p_l^{\max} + p_{l-1}^{\min}s + p_{l-2}^{\min}s^2 + p_{l-3}^{\max}s^3 + p_{l-4}^{\max}s^4 + p_{l-5}^{\min}s^5$$
$$+ p_{l-6}^{\min}s^6 + p_{l-7}^{\max}s^7 + \cdots$$

$$d_K^4(s) = p_l^{\max} + p_{l-1}^{\max}s + p_{l-2}^{\min}s^2 + p_{l-3}^{\min}s^3 + p_{l-4}^{\max}s^4 + p_{l-5}^{\max}s^5$$
$$+ p_{l-6}^{\min}s^6 + p_{l-7}^{\min}s^7 + \cdots \qquad (13)$$

With the Kharitonov polynomials defined in (13), form the following sets of Kharitonov segments

$$\mathcal{S}_N = \left\{ n_S^{12}(s), n_S^{13}(s), n_S^{24}(s), n_S^{34}(s) \right\} \text{ and}$$

$$\mathcal{S}_D = \left\{ d_S^{12}(s), d_S^{13}(s), d_S^{24}(s), d_S^{34}(s) \right\}$$

where $n_S^{ij}(s) = (1 - \lambda)n_K^i(s) + \lambda n_K^j(s)$ and $d_S^{ij}(s) = (1 - \lambda) d_K^i(s) + \lambda d_K^j(s)$, $0 \le \lambda \le 1$. Finally, the 32 generalised Kharitonov segments are formed as follows

$$\begin{cases} \delta_{S_D}^{ijk}(s) = n_G(s)n_K^i(s) + d_G(s)d_S^{jk}(s), \\ \quad i = 1, 2, 3, 4 \text{ and } jk \text{ defined in } \mathcal{S}_D \\ \delta_{S_N}^{jki}(s) = n_G(s)n_S^{jk}(s) + d_G(s)d_K^i(s), \\ \quad jk \text{ defined in } \mathcal{S}_N \text{ and } i = 1, 2, 3, 4 \end{cases} \qquad (14)$$

The following result can then be stated:

Theorem 2: For a given $\tilde{p}$, all the closed-loop characteristic polynomials formed with all possible controllers whose coefficients lie in the intervals defined in (9) are Hurwitz if and only if all the 32 Generalised Kharitonov segments defined in (14) are Hurwitz.

Proof: See [2, p. 300]. □

Remark 3: An easy way to check the stability of the 32 generalised Kharitonov segments defined in (14) is provided by the so-called bounded phase lemma [2, p. 72]. In accordance with the bounded phase lemma, given two Hurwitz stable polynomials $\delta_1(s)$ nd $\delta_2(s)$ of degree n, and assuming that the polynomial segment $\delta^{12}(s) = (1 - \lambda)\delta_1(s) + \lambda\delta_2(s)$ has degree n for all $\lambda \in [0, 1]$, then $\delta^{12}(s)$ is stable if and only if $|\phi_1(j\omega) - \phi_2(j\omega)| \ne \pi$ rd for $\omega \in \mathbb{R}$, where $\phi_i(j\omega)$ denotes the phase of $\delta_i(j\omega)$.
In this paper, the smallest value of the perturbation $\tilde{\rho}$ in the controller coefficients that makes the closed-loop system unstable obtained by using the generalised Kharitonov theorem will be denoted by ρ_{GK}. The search for ρ_{GK} can be carried out according to the following algorithm.

Algorithm 2: Make $k = 1$ and choose a value for $\tilde{\rho}_1$.

Step 1. Compute the 32 generalised Kharitonov segments defined in (14).
Step 2. Use the bounded phase lemma to check the stability of each segment. If all the segments are stable, then make $k = k + 1$, choose $\tilde{\rho}_{k+1} > \tilde{\rho}_k$ and go back to Step 1. If at least one generalised Kharitonov segment has an unstable polynomial, then use bisection between ρ_k ρ_{k-1} to find the smallest value of $\tilde{\rho}$ (ρ_{GK}) for which at least one of the 32 generalised Kharitonov segments becomes unstable.

Remark 4: Since the values of ρ_{LP} and ρ_{GK}, obtained in Sections 3.1 and 3.2, respectively, are both obtained from necessary and sufficient conditions, they must be equal.

3.3 Comparative examples

To compare the results of this paper with those given previously in terms of the relative parametric stability margin, all the continuous-time domain examples considered in [1] are re-examined here. Table 1 shows the maximum allowed percent perturbation in each parameter of the coefficients of the numerator and denominator

Table 1: Maximum allowed perturbation in the controller coefficients

Example	$\bar{p}$(%)	ρ_{LP}(%), ρ_{GK}(%)	$\rho_{LP}/\bar{p}$, $\rho_{GK}/\bar{p}$
1	2.1034×10^{-5}	0.0377	1792.3
1*	8.5770×10^{-5}	0.14	1632.3
2	7.21	11.17	1.5492
3	1.16	4.34	3.7414
4	1.4553×10^{-13}	6.58	4.5214×10^{13}
6	3.7370×10^{-4}	0.98	2622.4

Fig. 1 *Nyquist diagram of $G(s)K_{3rd}(s)$ (solid line) and $G(s)K_{1st}(s)$ (dash-dotted line)*

polynomials given by the relative parametric stability margin (first column) and by the non-conservative measures proposed in this paper (second column); the ratio between the aforementioned measures are given in the third column. Notice that, even for the notorious Example 1 of [1], there is a significant change in the maximum allowed perturbation: the relative parametric stability margin only allows a perturbation of up to 2.1034×10^{-5}% whereas using the parametric stability margins proposed in this paper, the controller parameters are guaranteed to face a change of up to 0.0377% without destabilising the closed-loop system, that is, 1792.3 times the value given by the relative parametric stability margin. The results are even better if the coefficients of the controller transfer function of Example 1 of [1] are replaced with those given in [5] (Example 1*). In this case, the measures proposed here allows a perturbation in the controller coefficients of $\sim$0.14% against only 8.577×10^{-5}% guaranteed by the relative parametric stability margin. Finally, notice, according to the fifth row of Table 1 (Example 4 of [1] and re-examined in Section 2), how conservative Keel and Battacharyya's stability margin can be. The compensated system can actually tolerate a change of up to 6.58% in the controller coefficients rather than the 1.45×10^{-13}% maximum change prescribed by the relative parametric stability margin. It is an improvement over the relative parametric stability margin by a factor of about 10^{13}!

4 Are H_{∞} robust controllers actually fragile?

In this section, the simple examples used in [1] to label H_{∞} controller as fragile are re-examined.

4.1 Example 1 of Keel and Bhattacharyya

This example was taken from [19]. The plant transfer function is given as

$$G(s) = \frac{s-1}{s^2 - s - 2} \quad (15)$$

and the design objective is to optimise the upper gain margin. The controller obtained via Q-parameterisations using robust control theory, to give an upper gain margin of 3.5 is

$$K(s) = \frac{\epsilon_0^0 s^6 + \epsilon_1^0 s^5 + \epsilon_2^0 s^4 + \epsilon_3^0 s^3 + \epsilon_4^0 s^2 + \epsilon_5^0 s + \epsilon_6^0}{\phi_0^0 s^7 + \phi_1^0 s^6 + \phi_2^0 s^5 + \phi_3^0 s^4 + \phi_4^0 s^3 + \phi_5^0 s^2 + \phi_6^0 s + \phi_7^0} \quad (16)$$

where $\epsilon_0^0 = 379$, $\epsilon_1^0 = 39\,383$, $\epsilon_2^0 = 192\,306$, $\epsilon_3^0 = 382\,993$, $\epsilon_4^0 = 383\,284$, $\epsilon_5^0 = 192\,175$ and $\epsilon_6^0 = 38\,582$, and $\phi_0^0 = 3$, $\phi_1^0 = -328$, $\phi_2^0 = -38\,048$, $\phi_3^0 = -179\,760$, $\phi_4^0 = -314\,330$, $\phi_5^0 = -239\,911$ and $\phi_6^0 = -67\,626$. The lower gain margin for this controller is [1, 0.9992] and the phase margin is [0, 0.1681] degrees. This shows that small
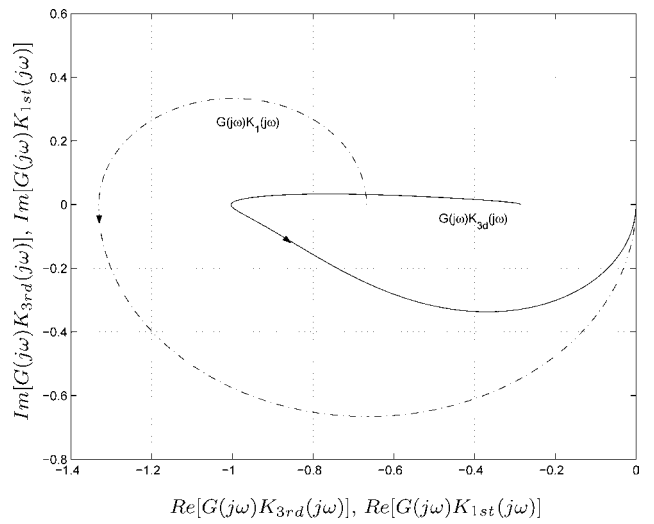
variations on the gain or phase can make the closed-loop system unstable. In [5], this example is re-examined, and it is shown that the controller computed in [1] and given by (16) is wrong, since an inaccurate $Q(s)$ was deployed in its computation. A correct sixth-order controller has been computed and because of the three common stable factors in the denominator and numerator, its transfer function has been reduced to a third-order one, given as

$$K_{3rd}(s)$$
$$= \frac{149.97 s^3 + 15\,301.13916 s^2 + 30\,568.08516 s + 15\,416.916}{s^3 - 133.375 s^2 - 14\,821.16945 s - 27\,027.939519} \quad (17)$$

However, this controller also leads to a poor lower gain margin [1, 0.9965] as it can be seen from the Nyquist diagram shown in Fig. 1 (solid line).

The explanation for the poor lower gain margin is as follows: the plant and controller have one unstable pole each, and thus, for closed-loop stability, it is necessary that the Nyquist diagram encircles the critical point $-1 + j0$ twice in a counterclockwise direction. Since $G(s)$ has relative degree equal to one, then the maximum gain margin will be given for $\omega = 0$, and therefore in order to maximise the upper gain margin, the Nyquist diagram shown in Fig. 1 must be shifted to the right, reducing therefore the lower gain margin. As a consequence, $K_{3rd}(s)$ can tolerate only 0.14% of variations in its parameters, as seen in Table 1, second row. It can therefore be concluded that the low tolerance to controller parameter variation, in this example, is due to the use of a bad optimisation criterion.

In [20], it has been suggested that the correct approach to this problem is the maximisation of the so-called gain-phase margin [21], which can be formulated as the H_{∞} problem min $\|S\|_{\infty}$, where $S(s)$ denotes the sensitivity function, leading to the following first-order controller

$$K_{1st}(s) = \frac{4027 s + 4037.5}{s - 3023} \quad (18)$$

This controller leads to better gain and phase margins than $K_{3rd}(s)$ as shown Fig. 1 (dash-dotted line). Indeed, the gain margin is [0.7512, 1.4975] and the phase margin is [0, -19.2185] degrees. Therefore computing the parametric stability margin either via linear programming or using

the generalised Kharitonov theorem, one obtains $\rho_{LP} = \rho_{GK} = 0.142$, which shows that the coefficients of $K_{1st}(s)$ can be subjected to perturbations up to 14.2% without destabilising the closed-loop system.

It is important to remark that although the replacement of the problem of maximisation of the upper gain margin, as done in [1, 19]($\min\|T\|_\infty$), with the problem of maximisation of the gain-phase margin, ($\min\|S\|_\infty$), as proposed here, has led to a satisfactory result, as far as controller fragility is concerned, in practice other control objectives such as stability robustness, noise attenuation, disturbance rejection and control signal limitation must be considered. In addition, the controller must satisfy some robust performance conditions in the presence of plant uncertainty, and therefore it becomes necessary to formulate 2-block H_∞ problems. In [6], it is demonstrated, via examples, that failure to include external disturbance and specifications on the control signal in the optimal control problem formulation may lead to serious problems in control systems design. It is also pointed out in [6, 22] that Glover–McFarlane loopshaping design method [23] can be used as a reasonable starting point for robust control design.

4.2 Example 2 of Keel and Bhattacharyya

To compare the performance of the optimal controller given by (16) with a non-optimal controller, an arbitrary first-order controller with transfer function

$$K_a(s) = \frac{11.44974739s + 11.24264066}{s - 7.03553383} \quad (19)$$

has been designed in [1]. Although, this controller has not been designed to maximize the upper gain margin, its gain and phase margins, [0.7940, 1.2516] and [0, −9.8873], respectively, were compared with those of the optimal controller computed in Example 1 given by (16), and it was concluded in [1] that the arbitrary controller is far less fragile than the optimal one. However, note that $K_a(s)$ leads to worse gain and phase margins than the H_∞ robust controller $K_{1st}(s)$ given by (18). In addition, $K_a(s)$ can tolerate less variations in its coefficients than $K_{1st}(s)$. Therefore, $K_{1st}(s)$ is less fragile than $K_a(s)$ on all counts.

Consider now the problem of maximising the closed-loop system tolerance to additive uncertainty in the plant transfer function, formulated as $\min\|KS\|_\infty$. The solution to this problem is given by

$$K_o(s) = \frac{12s + 12}{s - 7} \quad (20)$$

Note that $K_o(s)$ has all its coefficients close to the coefficients of $K_a(s)$, which suggests that the arbitrary controller $K_a(s)$ can be seen as a suboptimal controller for the problem $\min\|KS\|_\infty$; actually $\|K_oS\|_\infty = 12$, whereas $\|K_aS\|_\infty = 14.9805$. It is worth remarking that the gain and phase margins for $K_o(s)$ are, respectively, [0.75, 1.1667] and [0, −10.2250] degree and that $K_o(s)$ can tolerate a change in its coefficients up to 7.69%. Therefore the H_∞ controller $K_o(s)$ is non-fragile.

4.3 Example 3 of Keel and Bhattacharyya

In this example, an optimal H_∞ robust controller has been designed with the view to minimising $\|WT\|_\infty$, where $T(s)$ is the complementary sensitivity function and $W(s)$ is a weighting function given by

$$W(s) = \frac{s + 0.1}{s + 1} \quad (21)$$

The plant and the robust controller transfer functions are, respectively, given as

$$G(s) = \frac{s - 1}{s^2 + 0.5s - 0.5},$$
$$K(s) = \frac{-124.5s^3 - 364.95s^2 - 360.45s - 120}{s^3 + 227.1s^2 + 440.7s + 220} \quad (22)$$

According to [1], this controller can only tolerate a change in its coefficients of at most 1.116%. However, this is not true, as one can see from Table 1: $K(s)$ can actually tolerate 4.34%. Therefore this controller is far less fragile than it was suggested in [1]. On the other hand, the system gain and phase margins are, respectively, [0.9166, 1.8051] and [0, 12.9112], which implies that this controller cannot be implemented in a real system since its lower gain margin is approximately equal to one. It is therefore necessary to improve the lower gain margin.

Notice that the closed-loop poles for $K(s)$ are -100, $-1.000005427911 + j0.000009401696$, $-1.000005427911 - j0.000009401696$, -0.999989144178 and -0.1, which reveals that three of them are very close to -1. It is well known that the polynomial zero multiplicity is a factor of sensitivity in the computation of polynomial zeros. Following this clue, Whidborne et al. [7] investigated the relationship between closed-loop pole multiplicity and controller fragility, and Mäkilä [5] suggests that this problem is part of the puzzle. However, the fragility in this example is not related to the repeated poles of the closed-loop system but to the zero of the weighting function $W(s)$.

To show this, note that the pole of the closed-loop system, closest to the origin, is equal to -0.1. It is well known that the closer the closed-loop poles are to the jw-axis, the closer the Nyquist diagram will be to the critical point $-1 + j0$. Notice also that this closed-loop pole is exactly the zero of the weighting function $W(s)$ and, as it can be proved (see Appendix), this always happens when a H_∞ robust controller is designed by optimally selecting the parameter $Q(s)$ of the Youla–Kucera parametrisation to solve 1-block problems. Therefore simply by changing appropriately the weighting function, one can improve the gain and phase margins. For example, choosing a new weighting function

$$W_2(s) = \frac{s + 0.2}{s + 1} \quad (23)$$

then the resulting H_∞ controller that minimises $\|W_2T\|_\infty$ is given by

$$K_2(s) = \frac{-144.5s^3 - 425.4s - 420.9s - 140}{s^3 + 247.2s^2 + 481.4s + 240} \quad (24)$$

The closed-loop poles for the system with $K_2(s)$ are -99.99999999999992, $-1.00000591959048 + j0.00001025331231$, $-1.00000591959048 - j0.00001025331231$, -0.99998816081903 and -0.2. Notice that the dominant pole is now -0.2 and therefore as expected, this system has better gain and phase margins, [0.8571, 1.6943] and [0, 16.2230] degrees, respectively. In addition, it can also be verified that $K_2(s)$ tolerates a change in its coefficients of up to 7.69%, which shows that $K_2(s)$ is less fragile than $K(s)$.

5 Conclusions

In this paper, it is shown that the relative parametric stability margin used by Keel and Bhattacharyya to label, as fragile, optimal and robust controllers can be very conservative when perturbations of a given polytopic nature are considered in the controller coefficients. With the view of showing this fact, two non-conservative measures have been proposed: a first one, based on the solution of a linear programming problem, and a second one, using the generalised Kharitonov theorem. Both measures have been applied to the same examples presented in the literature to illustrate the fragility problem, and it is shown that the controllers, once labelled as fragile, can actually tolerate much larger perturbations in its coefficients than that prescribed by the relative parametric stability margin.

As far as the alleged fragility of H_∞ controller, all the examples presented in [1] are re-examined here, leading to the following conclusions: (i) the H_∞ optimisation in Example 1 of [1] (optimisation of the upper gain margin) is badly formulated and is suggested here that the correct formulation is via optimisation of the so-called gain-phase margin; (ii) the arbitrary controller of Example 2 of [1] is actually a suboptimal controller for the H_∞ minimisation problem $\min \|KS\|_\infty$; (iii) the lower gain margin and the relative parametric stability margin for the system of Example 3 can be improved simply by choosing another weighting function with zeros farther to the left in the s-plane.

6 Acknowledgment

7 References

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8 Appendix

In this appendix, the synthesis of H_∞ robust controllers [19] for 1-block problems is briefly reviewed and it is shown that when the optimisation problem is formulated using a weighting function, then the zero of this function is a pole of the closed-loop system.

According to the H_∞ theory, the optimal controller is designed via Youla–Kucera parametrisation, that is

$$K(s) = [Y(s) + M(s)Q(s)][X(s) - N(s)Q(s)]^{-1} \qquad (25)$$

where $Q(s) \in \mathbb{R}H_\infty$ is the parameter to be optimally selected, and $G(s) = N(s)M^{-1}(s)$, with $N(s)$, $M(s)$, $X(s)$ $Y(s) \in \mathbb{R}H_\infty$ satisfying the Bezout identity

$$X(s)M(s) + Y(s)N(s) = 1 \qquad (26)$$

For plant models given in terms of strictly proper transfer functions, the H_∞ problems $\min \|WS\|_\infty$, $\min \|WSG\|_\infty$ and $\min \|WT\|_\infty$ with non-trivial solution, where $S(s)$ and $T(s)$ are the sensitivity and complementary sensitivity functions, respectively, and $W(s)$ is the weighting function, do not have a proper solution, that is, $Q(s)$ solution of the H_∞ problem is improper. Consequently, the optimal controllers obtained by substituting the optimal $Q(s)$ in the parametrisation given in (25) will also be improper. To circumvent this problem, $Q(s)$ is divided by a factor $(\tau s + 1)^\nu$ to make it proper, where τ is chosen to be suitably small and positive.

A systematic way to compute the optimal $Q(s)$ is presented in [24], consisting basically of the following steps:

1. write the optimisation problem as a model-matching problem $\min \|T_1 - T_2 Q\|_\infty$;
2. compute the inner–outer factorisation of $T_2(s)$, that is

$$T_2(s) = T_{2_i}(s)T_{2_o}(s) \qquad (27)$$

3. write

$$\|T_1 - T_2 Q\|_\infty = \|R - X_S\|_\infty \qquad (28)$$

where $R(s) = T_{2_i}^\star(s)T_1(s)$.
4. find $X_S(s) \in \mathbb{R}H_\infty$ that minimises $\|R - X_S\|_\infty$.

5. compute the optimal $Q(s)$

$$Q(s) = T_{2_o}^{-1}(s)X_S(s). \qquad (29)$$

It is not difficult to prove that, for all 1-block problems, the zeros of the weighting function $W(s)$ are also zeros of $T_{2_o}(s)$ [20]. Therefore according to (29), these zeros are also poles of $Q(s)$. To show that this fact implies that the zeros of $W(s)$ are also closed-loop poles, it is necessary first to write the transfer functions of $N(s)$ and $M(s)$ as

$$N(s) = \frac{n_N(s)}{d(s)} \quad \text{and} \quad M(s) = \frac{n_M(s)}{d(s)} \qquad (30)$$

where $n_N(s)$ and $n_M(s)$ are, respectively, the numerator and denominator of $G(s)$ and $d(s)$ is a Hurwitz polynomial. In addition, write $X(s)$ and $Y(s)$ as

$$X(s) = \frac{n_X(s)}{\hat{d}(s)} \quad \text{and} \quad Y(s) = \frac{n_Y(s)}{\hat{d}(s)} \qquad (31)$$

where $n_X(s)$ and $n_Y(s)$ are the numerators of $X(s)$ and $Y(s)$, respectively, and $\hat{d}(s)$ is a Hurwitz polynomial. Notice that $d(s)$ and $\hat{d}(s)$ can always be chosen to be equal [25] and therefore without loss of generality, in this paper $d(s) = \hat{d}(s)$. Making

$$K(s) = \frac{n_K(s)}{d_K(s)} \qquad (32)$$

and, in the sequel, substituting $N(s)$, $M(s)$, $X(s)$, $Y(s)$ and $K(s)$, according to (30–32), in (25), it is possible, after simple algebraic manipulations, to write $Q(s)$ as

$$Q(s) = \frac{n_K(s)n_X(s) - n_Y(s)d_K(s)}{n_K(s)n_N(s) + d_K(s)n_M(s)} \qquad (33)$$

Notice that the denominator of $Q(s)$ is exactly the closed-loop characteristic polynomial and therefore this shows that the zeros of $W(s)$ are also poles of the closed-loop system.