Fair and Square Computation of Inverse $\mathcal{Z}$-Transforms of Rational Functions

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***Abstract*—All methods presented in textbooks for computing inverse $\mathcal{Z}$-transforms of rational functions have some limitation: 1) the direct division method does not, in general, provide enough information to derive an analytical expression for the time-domain sequence $x(k)$ whose $\mathcal{Z}$-transform is $X(z)$; 2) computation using the inversion integral method becomes labored when $X(z)z^{k-1}$ has poles at the origin of the complex plane; 3) the partial-fraction expansion method, in spite of being acknowledged as the simplest and easiest one to compute the inverse $\mathcal{Z}$-transform and being widely used in textbooks, lacks a standard procedure like its inverse Laplace transform counterpart. This paper addresses all the difficulties of the existing methods for computing inverse $\mathcal{Z}$-transforms of rational functions, presents an easy and straightforward way to overcome the limitation of the inversion integral method when $X(z)z^{k-1}$ has poles at the origin, and derives five expressions for the pairs of time-domain sequences and corresponding $\mathcal{Z}$-transforms that are actually needed in the computation of inverse $\mathcal{Z}$-transform using partial-fraction expansion.**

***Index Terms*—Control education, discrete-time signals, discrete-time systems, inverse $\mathcal{Z}$-transformation, teaching methodology.**

I. Introduction

THE $\mathcal{Z}$-transform is an important mathematical tool and plays a key role in the analysis and design of discrete-time systems. It is usually taught as part of a discrete-time control course in electrical and electronic engineering curricula. The $\mathcal{Z}$-transform is a transformation that maps discrete-time signals to complex rational functions, being defined as follows:

$$\begin{aligned}\mathcal{Z} : \mathbb{Z} \times \mathbb{R} &\rightarrow \mathbb{C} \times \mathbb{C} \\ x(k) &\mapsto X(z) = \mathcal{Z}[x(k)] = \sum_{0}^{\infty} x(k) z^{-k}. \end{aligned} \qquad (1)$$

It is usually obtained by using convergence properties of complex series [1], [2].

Taking into account the fact that signals considered in engineering usually have rational $\mathcal{Z}$-transforms and the concept of transfer function, it can be seen that the output of the system in the time-domain can be obtained by simply computing the inverse $\mathcal{Z}$-transform of the product of the discrete transfer function of the system and the $\mathcal{Z}$-transform of the input signal. Therefore, the computation of inverse $\mathcal{Z}$-transforms is crucial in the performance analysis of discrete linear time-invariant systems.

The computation of inverse $\mathcal{Z}$-transforms is performed by means of one of the following three methods [3]–[10]:

Z1) the direct division method;
Z2) the inversion integral method;
Z3) the partial-fraction expansion method.

The direct division method Z1 is a straightforward way to obtain the time-domain sequence $x(k)$ whose $\mathcal{Z}$-transform is $X(z)$. However, this method is only suitable when it is necessary to know the first terms of the time-domain sequence since, apart from very special cases, it is not possible to obtain an analytic expression for $x(k)$.

The inversion integral method Z2 relies on the computation of the following integral:

$$x(k) = \frac{1}{2\pi j} \oint_{\mathcal{C}} X(z) z^{k-1} dz \qquad (2)$$

where $\mathcal{C}$ is a counterclockwise contour that encloses the origin and all poles of $X(z)z^{k-1}$. The solution to this integral requires some knowledge of complex variable theory, with the sequence $x(k)$ being obtained by using the residue theorem. However, as pointed out in [6], the computation of residues becomes cumbersome when $X(z)z^{k-1}$ has poles at the origin of the complex plane. This difficulty has been partially circumvented in [8] by using the change of variable $z = 1/u$. It can be easily checked that for the values of k for which $X(1/u)u^{-k+1}$ has no poles at the origin, then $x(k) = 0$. However, it is still necessary to compute the residues at the origin corresponding to the values of k not encompassed by the analytic expression for $x(k)$ that results from the computation of the residues of $X(z)z^{k-1}$ associated with the poles that are not at the origin. Therefore, although the approach proposed in [8] reduces the awkwardness pointed out in [6], obtaining the complete solution is still labored.

The partial-fraction expansion method Z3 is considered by some authors [3], [4], [7] as preferable to the inversion integral method, being regarded as the simplest and easiest way to compute the inverse $\mathcal{Z}$-transform of a rational function $X(z)$. In spite of being widely used in textbooks, the proposed procedures differ from author to author, and the tables of $\mathcal{Z}$-transform pairs, when given, are both long and incomplete. The lack of an expression for the time-domain sequence that corresponds to the term with multiple poles is without a doubt the main reason for the number of different approaches to compute inverse $\mathcal{Z}$-transforms using partial-fraction expansion. Franklin *et al.* [5] restrict themselves to the direct division and inversion integral methods. Jackson [10] and Mitra [9] address this problem in

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a very superficial way. Cadzow and Martens [3] take into account poles with multiplicity less than or equal to five, propose a partial-fraction expansion in which the numerator of the terms of degree greater than one are polynomials (not a constant), and present a table with the pairs of the terms considered. Oppenheim and Schafer [4] present a partial-fraction expansion in terms of negative powers of z, but do not specifically address the inversion of the terms with degree different from one. Soliman and Srinath [7] suggest the use of a 20-row table to identify the sequences corresponding to the terms in the partial fraction expansion. Ogata's book [6] restricts its treatment to multiplicity two and recommends that for a triple pole at $z = z_i$, the numerator of the term $(z - z_i)^3$ must include a term $(z + z_i)$. Multiplicities larger than three are not considered in [6].

This paper addresses the computation of the inverse $\mathcal{Z}$-transform of a proper rational function $X(z)$ and proposes a simple remedy to overcome the alleged cumbersomeness of the use of the inversion integral method Z2 when the function $X(z)z^{k-1}$ has poles at the origin for some values of k. In addition, a list with only five pairs of time-domain sequences/$\mathcal{Z}$-transforms for all the terms that are actually needed in the computation of inverse $\mathcal{Z}$-transforms using partial-fraction expansion is derived here.

This paper is organized as follows. The computation of inverse $\mathcal{Z}$-transform is considered in Section II, addressing all methods Z1–Z3, highlighting their importance and dependence on each other. In Section III, two examples are used to illustrate this paper's contributions. Student assessment is reported in Section IV. Finally, conclusions are drawn in Section V.

II. Computation of Inverse $\mathcal{Z}$-Transform

Let the $\mathcal{Z}$-transform of a real sequence $x(k)$, obtained according to (1), be expressed as

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_m}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n} \tag{3}$$

where $b_j,\ a_i \in \mathbb{R}$, for $j = 0, \ldots, m$ and $i = 1, \ldots, n$, and $n \geq m$. Assuming that $X(z)$ has n_0 ($n_0 \geq 0$) poles at the origin of the complex plane, then the last n_0 coefficients of the denominator of $X(z)$, namely, $a_{n-n_0+1}, a_{n-n_0+2}, \ldots, a_n$, are identically zero, and thus (3) can be rewritten as

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_m}{z^{n_0}\left(z^{n-n_0} + a_1 z^{n-n_0-1} + \cdots + a_{n-n_0}\right)}. \tag{4}$$

Methods Z1–Z3 for computing $x(k) = \mathcal{Z}^{-1}[X(z)]$ will be now considered.

A. Direct Division Method

Since, by assumption, $n \geq m$, then dividing the numerator and denominator polynomials of $X(z)$, given in (4), results in

$$X(z) = \frac{b_0 z^{m-n} + b_1 z^{m-n-1} + b_2 z^{m-n-2} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n-n_0} z^{n_0-n}}. \tag{5}$$

Defining $r = n - m$, then (5) can also be written as

$$X(z) = \frac{b_0 z^{-r} + b_1 z^{-r-1} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n-n_0} z^{n_0-n}}. \tag{6}$$

It can immediately be seen that $X(z)$ can be expanded into an infinite power series in z^{-1}. Therefore, according to (1), the terms $x(k)$ of the time-domain sequence will be the coefficients of z^{-k}, $k = 0, 1, 2, \ldots$. The following conclusions can be drawn.

C1) The first r terms of the sequence $x(k)$ ($x(0)$ to $x(r-1)$) are equal to zero.

C2) $x(r) = b_0$.

C3) The remaining terms of $x(k)$ (i.e., $k > r$) can be obtained through straightforward polynomial division.

It is not hard to see that, in general, this method does not provide the means to obtain a closed-form expression for $x(k)$ and is thus suitable only when it is necessary to obtain the first few terms of $x(k)$. Nevertheless, as will be seen in Section II-B, this fact will be used in order to overcome the limitation of the inversion integral method when $X(z)z^{k-1}$ has poles at the origin.

B. Inversion Integral Method

Another way to compute the inverse $\mathcal{Z}$-transform of a proper rational function is by using the Cauchy integral theorem, leading to the inversion integral given in (2). One way to evaluate the contour integral (2) is through the Cauchy's residue theorem [1], which states that if $X(z)z^{k-1}$ has p distinct poles inside $\mathcal{C}$, then $x(k)$ will be given by

$$x(k) = \sum_{i=1}^{p} \operatorname{Res}_{z=z_i}\left[X(z)z^{k-1}\right] \tag{7}$$

where $\operatorname{Res}_{z=z_i}(.)$ denotes the residue at pole z_i. If z_i is a simple pole, then

$$\operatorname{Res}_{z=z_i}\left[X(z)z^{k-1}\right] = \lim_{z\to z_i}\left[(z - z_i)X(z)z^{k-1}\right] \tag{8}$$

and when z_i is a pole of multiplicity q, then

$$\begin{aligned}\operatorname{Res}_{z=z_i}&\left[X(z)z^{k-1}\right]\\ &= \frac{1}{(q-1)!}\lim_{z\to z_i}\frac{d^{q-1}}{dz^{q-1}}\left[(z - z_i)^q X(z)z^{k-1}\right].\end{aligned} \tag{9}$$

It is remarked in [6] that when $X(z)z^{k-1}$ has poles at the origin for some values of k, then the application of the inversion integral method becomes cumbersome. The explanation of this follows. Assume that the numerator and denominator of $X(z)$ are coprime and that $X(z)$ has no zeros at the origin. Since, by assumption, $X(z)$ has n_0 poles at the origin, then $X(z)$ can be written as

$$X(z) = \frac{1}{z^{n_0}}\tilde{X}(z) \tag{10}$$

where $\tilde{X}(z)$ has no poles at the origin. Thus

$$X(z)z^{k-1} = \frac{1}{z^{n_0-k+1}}\tilde{X}(z) \tag{11}$$

has $n_{0,k} = n_0 + 1 - k$ poles at the origin, which implies that the number of poles of $X(z)z^{k-1}$ at the origin, $n_{0,k}$, depends on the value of k. Assuming that $\tilde{X}(z)$ has $\tilde{p}$ distinct poles $z_1, z_2, \ldots, z_{\tilde{p}}$, then $x(k)$, $k = 0, \ldots, n_0$, will be given as

$$x(0) = \mathrm{Res}_{z=0}\left[\frac{\tilde{X}(z)}{z^{n_0-1}}\right] + \sum_{i=1}^{\tilde{p}} \mathrm{Res}_{z=z_i}\left[\frac{\tilde{X}(z)}{z^{n_0-1}}\right] \tag{12}$$

$$\vdots$$

$$x(n_0) = \mathrm{Res}_{z=0}\left[\frac{\tilde{X}(z)}{z}\right] + \sum_{i=1}^{\tilde{p}} \mathrm{Res}_{z=z_i}\left[\frac{\tilde{X}(z)}{z}\right] \tag{13}$$

and for $k \geq n_0 + 1$

$$x(k) = \sum_{i=1}^{\tilde{p}} \mathrm{Res}_{z=z_i}\left[\frac{\tilde{X}(z)}{z^{n_0-k+1}}\right]. \tag{14}$$

It is therefore not difficult to see that it is necessary to compute $(n_0+1) \times (\tilde{p}+1)$ residues in order to calculate the first n_0+1 terms of the sequence. Looking at the problem from this perspective, it is not difficult to agree with [6] that the computation becomes cumbersome. However, it is also clear from (12) and (13) that the first $n_0 + 1$ terms do not depend on k. This observation provides a simple way to overcome the alleged cumbersomeness: Instead of computing $x(k)$ for each $k = 0, 1, \ldots, n_0$ using (7), it is enough to deploy the direct division method to obtain the first $n_0 + 1$ terms of the sequence, and then use (14) to calculate $x(k)$ for $k \geq n_0 + 1$.

It is important note that if the relative degree[1] r of $X(z)$ is greater than or equal to the number of poles n_0 of $X(z)$ at the origin, then it is not necessary to use the direct division method since, according to conclusions C1 and C2 of Section II-A, the first $n_0 + 1$ terms of $x(k)$ can be readily obtained by inspection of $X(z)$ when written in the form of (6).

C. Partial-Fraction Expansion Method

Let $X(z)$ be given by (3). In order to find the inverse $\mathcal{Z}$-transform of $X(z)$ through partial-fraction expansion, the first step is to compute the partial-fraction expansion of $X(z)/z$; the reason for which will become clear in what follows. Assume that $X(z)/z$ has p distinct poles $z_i \in \mathbb{C}$, each one with multiplicity q_i ($q_1 + q_2 + \cdots + q_p = n$). Since $m < n$, then $X(z)/z$ can be expanded as

$$\begin{aligned}\frac{X(z)}{z} &= \frac{A_{11}}{(z-z_1)} + \frac{A_{12}}{(z-z_1)^2} + \cdots + \frac{A_{1q_1}}{(z-z_1)^{q_1}} + \cdots \\ &\quad + \frac{A_{p_1}}{(z-z_p)} + \frac{A_{p_2}}{(z-z_p)^2} + \cdots + \frac{A_{pq_p}}{(z-z_p)^{q_p}}.\end{aligned} \tag{15}$$

[1]The relative degree of a rational function is the difference between the degrees of its numerator and denominator polynomials.

When z_i is a simple pole, then $q_i = 1$, and the only coefficient to be computed is A_{i1}, which is given by

$$A_{i1} = \lim_{z\to z_i}(z - z_i)\frac{X(z)}{z}. \tag{16}$$

If z_i is a multiple pole, then the coefficients A_{ij} are computed as follows:

$$A_{ij} = \lim_{z\to z_i} \frac{1}{(q_i - j)!}\frac{d^{q_i-j}}{dz^{q_i-j}}\left[(z - z_i)^{q_i}\frac{X(z)}{z}\right], \quad j = 1, \ldots, q_i. \tag{17}$$

Multiplying both sides of (15) by z gives

$$\begin{aligned}X(z) &= \frac{A_{11}z}{(z-z_1)} + \frac{A_{12}z}{(z-z_1)^2} + \cdots + \frac{A_{1q_1}z}{(z-z_1)^{q_1}} + \cdots \\ &\quad + \frac{A_{p_1}z}{(z-z_p)} + \frac{A_{p_2}z}{(z-z_p)^2} + \cdots + \frac{A_{pq_p}z}{(z-z_p)^{q_p}}.\end{aligned} \tag{18}$$

Thus, to compute the inverse $\mathcal{Z}$-transform of $X(z)$, it is only necessary to know is the inverse $\mathcal{Z}$-transform of the terms on the right-hand side of (18), which can be arranged in five types, as follows: 1) single or multiple poles at the origin: $z_i = 0$, $q_i \geq 1$; 2) a single real pole: $z_i = a$ ($a \in \mathbb{R}^*$), $q_i = 1$; 3) a single complex pole: $z_i = ae^{j\omega_0}$ ($a \in \mathbb{R}_+^*$), $q_i = 1$; 4) multiple real poles: $z_i = a$, ($a \in \mathbb{R}^*$), $q_i > 1$; 5) multiple complex poles: $z_i = ae^{j\omega_0}$($a \in \mathbb{R}_+^*$), $q_i > 1$. These cases will now be addressed.

1) Terms Containing Single or Multiple Poles at the Origin: Using (1), the following pair can be easily obtained:

$$x(k) = \delta(k - n_0) \leftrightarrow X(z) = z^{-n_0} = \frac{1}{z^{n_0}}, \qquad n_0 \in \mathbb{N} \tag{19}$$

where $\delta(k - n_0) = 1$, for $k = n_0$, and zero elsewhere. This pair is the relationship sought in Case 1.

2) Terms Containing a Single Real Pole: Assuming that $a \in \mathbb{R}^*$, then using (1), it is not difficult to see that the solution to Case 2 is given by

$$x(k) = a^k,\ a \in \mathbb{R}^* \leftrightarrow X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a},\ |z| > |a|. \tag{20}$$

3) Terms Containing Single Complex Poles: If $z_i = ae^{j\omega_0}$ ($a \in \mathbb{R}_+^*$) is a complex pole of $X(z)$, so is $z_i^* = ae^{-j\omega_0}$. Therefore, the two terms below must appear in the partial-fraction expansion (18)

$$Y(z) = \frac{Ae^{j\phi}z}{(z - z_i)} + \frac{Ae^{-j\phi}z}{(z - z_i^*)} \tag{21}$$

where $A \in \mathbb{R}_+^*$. Using (20), one may write

$$y(k) = Ae^{j\phi}a^k e^{j\omega_0 k} + Ae^{-j\phi}a^k e^{-j\omega_0 k}, \qquad \text{for } k \geq 0. \tag{22}$$

Therefore, (22) can be rewritten as

$$\begin{aligned}y(k) &= Aa^k e^{j(\omega_0 k+\phi)} + Aa^k e^{-j(\omega_0 k+\phi)} \\ &= 2Aa^k \cos(\omega_0 k + \phi),\ k \geq 0\end{aligned} \tag{23}$$

solving, therefore, Case 3.

TABLE I
INVERSE $\mathcal{Z}$-TRANSFORM FOR THE GENERAL TERMS IN PARTIAL-FRACTION EXPANSION

Type of poles	Term	Inverse $\mathcal{Z}$-transform
Single/multiple poles at $z=0$	$\frac{A}{z^{n_0}}$	$A\delta(k-n_0)$
Single real pole	$\frac{Az}{z-a}$	$Aa^k, k\geq 0$
Multiple real pole	$\frac{Az}{(z-a)^q}$	$\frac{A}{(q-1)!}a^{k-q+1}\prod_{i=0}^{q-2}(k-i), k\geq 0$
Single complex poles	$\frac{Ae^{j\phi}z}{(z-ae^{j\omega_0})}+\frac{Ae^{-j\phi}z}{(z-ae^{-j\omega_0})}, A,a\in\mathbb{R}_+$	$2Aa^k\cos(\omega_0 k+\phi), k\geq 0$
Multiple complex poles	$\frac{Ae^{j\phi}z}{(z-ae^{j\omega_0})^q}+\frac{Ae^{-j\phi}z}{(z-ae^{-j\omega_0})^q}, A,a\in\mathbb{R}_+$	$2\frac{A}{(q-1)!}a^{k-q+1}\cos[\omega_0(k-q+1)+\phi]$ $\times\prod_{i=0}^{q-2}(k-i), k\geq 0$

4) Terms Containing Multiple Real Poles: Assume that $a \in \mathbb{R}^*$ and define

$$Y(z)=\frac{Az}{(z-a)^q}. \tag{24}$$

Using (9), the corresponding time-domain sequence $y(k)$ can be obtained, being given as

$$\begin{aligned} y(k) &= \mathrm{Res}_{z=a}\left[Y(z)z^{k-1}\right] && (25)\\ &= \frac{1}{(q-1)!}\lim_{z\to a}\frac{d^{q-1}}{dz^{q-1}}\left[(z-a)^q\frac{Az}{(z-a)^q}z^{k-1}\right] \\ &= \frac{A}{(q-1)!}a^{k-q+1}\prod_{i=0}^{q-2}(k-i), \qquad k\geq 0. && (26)\end{aligned}$$

5) Terms Containing Multiple Complex Poles: As in the case of single complex poles, the following expression must be considered:

$$Y(z)=\frac{Ae^{j\phi}z}{(z-ae^{j\omega_0})^q}+\frac{Ae^{-j\phi}z}{(z-ae^{-j\omega_0})^q}. \tag{27}$$

The inverse $\mathcal{Z}$-transform of each term on the right-hand side of (27) can be obtained using (25), which results in

$$\begin{aligned} y(k) = &\frac{A}{(q-1)!}\left[e^{j\phi}(ae^{j\omega_0})^{k-q+1}+e^{-j\phi}(ae^{-j\omega_0})^{k-q+1}\right] \\ &\times\prod_{i=0}^{q-2}(k-i), k\geq 0. \end{aligned} \tag{28}$$

Therefore, (28) can be rewritten as

$$\begin{aligned} y(k) = 2\frac{Aa^{k-q+1}}{(q-1)!}\cos\left[\omega_0(k-q+1)+\phi\right]\times\prod_{i=0}^{q-2}(k-i),\\ k\geq 0. \end{aligned} \tag{29}$$

The results of this section are summarized in Table I. Notice that it is enough to know only the five pairs given in Table I to compute the inverse $\mathcal{Z}$-transform of a rational function. This fact makes the teaching of inverse $\mathcal{Z}$-transforms using partial fraction expansion easy and straightforward since an analog table containing pairs of functions/Laplace transforms was previously derived when inverse Laplace transforms using partial fraction expansion was taught.

Remark 1: The usual approach to the computation of inverse $\mathcal{Z}$-transform in the case of multiple poles (real or complex) requires that the numerators of the partial fraction expansion terms be polynomials. Because of the way the denominator polynomials of the partial fraction expansion of $X(z)/z$ are chosen in the methodology proposed here, and due to the fact that (25) has been derived from (24) by using the inversion integral, the numerator of all terms of the partial fraction expansion of $X(z)$ are polynomials of degree 1 without the constant term. This relationship between the partial fraction expansion and the inversion integral is the key step to establishing the partial fraction expansion proposed here, which can be regarded as the inverse Laplace transform counterpart. Notice that the computation of inverse Laplace transforms using partial fraction expansion also does not require polynomial numerators. □

III. EXAMPLES

Two numerical examples are presented here to illustrate the methodology proposed in this paper.

A. Example 1

In this example, $x(k)=\mathcal{Z}^{-1}[X(z)]$ will be computed for

$$X(z)=\frac{z+2}{(z-2)z^2} \tag{30}$$

using the inversion integral method. Note that since

$$X(z)z^{k-1}=\frac{(z+2)z^{k-3}}{(z-2)} \tag{31}$$

then the multiplicity of the poles of $X(z)z^{k-1}$ at the origin depends on the value of k, as follows: 1) $k=0$: $z_1=0(\#3)$ and $z_2=2$; 2) $k=1$: $z_1=0(\#2)$ and $z_2=2$; 3) $k=2$: $z_1=0$ and $z_2=2$; 4) $k\geq 3$: $z_1=2$, where $(\#.)$ denotes multiplicity. This makes the task of computing $x(k)$ long and tedious since seven residues must be computed, as follows:

$$x(k)=\begin{cases} R_{01}+R_{02}, & k=0\\ R_{11}+R_{12}, & k=1\\ R_{21}+R_{22}, & k=2\\ R_3(k), & k\geq 3\end{cases}$$

where

$$R_{01}=\mathrm{Res}_{z=0}\left[\frac{(z+2)}{z^3(z-2)}\right]$$

$$\begin{aligned}
R_{02} &= \mathrm{Res}_{z=2}\left[\frac{(z+2)}{z^3(z-2)}\right]\\
R_{11} &= \mathrm{Res}_{z=0}\left[\frac{(z+2)}{z^2(z-2)}\right]\\
R_{12} &= \mathrm{Res}_{z=2}\left[\frac{(z+2)}{z^2(z-2)}\right]\\
R_{21} &= \mathrm{Res}_{z=0}\left[\frac{(z+2)}{z(z-2)}\right]\\
R_{22} &= \mathrm{Res}_{z=2}\left[\frac{(z+2)}{z(z-2)}\right]\\
R_3(k) &= \mathrm{Res}_{z=2}\left[\frac{(z+2)z^{k-3}}{(z-2)}\right].
\end{aligned}$$

Notice that, according to Section II-B, it is not necessary to compute the first six residues; terms $x(0)$, $x(1)$, and $x(2)$ can be obtained in a straightforward way (see conclusions C1 and C2 in Section II-A) as follows.

1) $x(0) = x(1) = 0$, since $X(z)$ has relative degree equal to 2.
2) $x(2) = 1$, the leading coefficient of the numerator polynomial of $X(z)$.

Therefore, the only residue that has to be computed is $R_3(k)$, which, according to (8), is given as

$$R_3(k) = \lim_{z\to 2}(z+2)z^{k-3} = 2^{k-1}, \qquad k \geq 3.$$

Thus

$$x(k) = \mathcal{Z}^{-1}[X(z)] = \begin{cases} 0, & k = 0, 1\\ 1, & k = 2\\ 2^{k-1}, & k \geq 3.\end{cases}$$

Remark 2: If the transformation proposed in [8] is performed in (31), then the following contour integral must be solved:

$$x(k) = \frac{1}{2\pi j}\oint_{\tilde{C}} \frac{u+0.5}{-u+0.5}u^{-k+1}d$$

where $\tilde{C}$ is any closed contour inside a circle of radius equal to 0.5. Since, due to the change of variable $z = 1/u$, the closed contour $\tilde{C}$ does not encircle the pole $u = 0.5$, it is not hard to see that for $k = 0, 1$, $x(k) = 0$. However, as shown in Example 1, $x(2)$ is also a constant. In order to obtain $x(2)$ using the contour integral above, it is still necessary to compute the residue for $k = 2$ as follows:

$$x(2) = \lim_{u\to 0}\frac{u+0.5}{-u+0.5} = 1$$

which is the same value as that easily obtained using the strategy proposed in this paper. □

B. Example 2

In this example, the inverse $\mathcal{Z}$-transform of

$$X(z) = \frac{z^{-3}}{(1-z^{-1})^3(1-z^{-1}+z^{-2})}$$

will be computed using the partial-fraction expansion method. According to Section II-C, the first step is to write $X(z)$ in positive powers of z, and then to obtain the partial-fraction expansion of $Y(z) = X(z)/z$. Since the poles of $X(z)$ are $z_1 = z_2 = z_3 = 1$, $z_4 = 1/2 + j\sqrt{3}/2 = e^{j(\pi/3)}$, and $z_5 = z_4^*$, then $Y(z)$ must be expanded as

$$Y(z) = \frac{A_1}{z-1} + \frac{A_2}{(z-1)^2} + \frac{A_3}{(z-1)^3} + \frac{B}{z-z_4} + \frac{B^*}{z-z_4^*} \quad (32)$$

where the coefficients A_1, A_2, A_3, and B are computed using (17) and (16) as follows:

$$\begin{aligned}
A_3 &= \lim_{z\to 1}(z-1)^3Y(z) = 1\\
A_2 &= \lim_{z\to 1}\frac{d}{dz}(z-1)^3Y(z) = 0\\
A_1 &= \lim_{z\to 1}\frac{1}{2}\frac{d}{dz}(z-1)^3Y(z) = -1\\
B &= \lim_{z\to z_4}(z-z_4)Y(z) = \frac{\sqrt{3}}{3}e^{-j(\pi/6)}.
\end{aligned}$$

Therefore, $X(z)$ has the following expansion:

$$X(z) = \frac{-z}{z-1} + \frac{z}{(z-1)^3} + \frac{\sqrt{3}}{3}\left(\frac{e^{-j(\pi/6)}z}{z-z_4} + \frac{e^{j(\pi/6)}z}{z-z_4^*}\right). \quad (33)$$

Using Table I on the right-hand side of (33) leads to the following expression:

$$x(k) = \frac{1}{2}\left(k^2-k-2\right) + \frac{2\sqrt{3}}{3}\cos\left(\frac{\pi k}{3} - \frac{\pi}{6}\right), \qquad k \geq 0.$$

IV. Student Assessment

In order to assess the methodology proposed in this paper, two surveys were carried out in the first and second semesters of 2010 with electrical engineering students attending their second control course. The students were asked to answer the questionnaire shown in Table II. Eighty-eight students (48 in the first semester and 40 in the second) answered the eight statements of the questionnaire by circling the grade that most closely described their opinion for each statement as follows: 5—absolutely agree; 4—mostly agree; 3—slightly agree; 2—slightly disagree; 1—mostly disagree; 0—absolutely disagree. Those students who scored 0 in statement 4 were asked not to score statements 5 and 7. Students were also told to score statement 8 only if they had discussed the method presented in the course with at least one student from other engineering courses.

Table III shows the results of the survey. Two conclusions can be drawn from the average scores shown in rows 1 and 2 of Table III: 1) the students had an above average background in Laplace transforms; and 2) partial-fraction expansion was their favorite method to perform the computation of inverse Laplace transforms. This fact had been realized by the authors in previous courses in which Laplace transforms were either taught or used as a tool. The opposite feeling was demonstrated by students when they were taught the computation of inverse $\mathcal{Z}$-transform with all the restrictions imposed in the textbooks cited in this paper. The students' disappointment was actually the motivation for seeking different approaches to deal with the computation of the inverse $\mathcal{Z}$-transform. The high average score of statement 3 presented in the third row of Table III shows the important role the inverse Laplace transform played in the process of understanding the computation method proposed here. The low average score of statement 4 reveals that

TABLE II
EIGHT STATEMENTS USED IN THE STUDENT SURVEY

1.	Before this course, I had a strong background in Laplace transforms. 5 4 3 2 1 0
2.	My favorite method to compute inverse Laplace transforms is the partial fraction expansion. 5 4 3 2 1 0
3.	My prior knowledge of how to calculate inverse Laplace transforms was crucial in helping me to understand the method presented in this course. 5 4 3 2 1 0
4.	I used other textbooks besides the lecture notes to study how to perform the computation of inverse $\mathcal{Z}$-transforms. 5 4 3 2 1 0
5.	I found the methods presented in the textbooks I used to complement my studies in computation of inverse $\mathcal{Z}$-transforms more appropriate than those taught in this course. 5 4 3 2 1 0
6.	I believe I will remember how to calculate inverse $\mathcal{Z}$-transforms even after the end of this course. 5 4 3 2 1 0
7.	I believe I would not be able to remember how to calculate inverse $\mathcal{Z}$-transforms in the future if I had been taught in accordance with the usual techniques presented in textbooks. 5 4 3 2 1 0
8.	In talking to other students from different engineering courses who had been taught inverse $\mathcal{Z}$-transforms according to standard textbooks I realized that my knowledge on this subject is much better than theirs. 5 4 3 2 1 0

TABLE III
RESULT OF SURVEY ANSWERED BY THE STUDENTS

Statement	Average score
1.	3.64
2.	4.36
3.	4.29
4.	1.24
5.	1.15
6.	3.67
7.	3.50
8.	4.10

only a few students sought other references than those given in class. Indeed, only 43% of the students consulted some textbook. It is worth remarking that the lecture notes on which this paper was based on were handed out to the students. Those students who browsed at least some textbooks found the method presented in class easier to understand than those presented in those textbooks, as demonstrated by the low average score associated with statement 5. Comparing the average scores for statements 1, 6, and 7, it can be verified that the students have approximately the same expectation of their knowledge of the $\mathcal{Z}$-transform as their evaluation of their own background in Laplace transforms. The result of statement 8 attests to the success achieved by the proposed method. The average score of statement 8 is very high (4.10), although only 34% of the students responded to this statement, which again shows that from the students' point of view, the method has superseded the standard ways to teach the inverse $\mathcal{Z}$-transform.

V. CONCLUSION

In this paper, simple and systematic ways to compute inverse $\mathcal{Z}$-transforms have been proposed. The teaching methodology presented here can be regarded as the inverse Laplace transform counterpart. A short but complete, list of pairs of time-domain sequences/$\mathcal{Z}$-transforms has been derived. An alleged cumbersomeness of the inversion integral method in the case of multiple poles at the origin has also been removed. Student assessment of the new teaching methodology proposed here has attested that the method can be effectively used in the teaching of inverse $\mathcal{Z}$ transforms.

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